10 NUMBER PATTERNS

LEARNING OBJECTIVES

In this topic, we will learn to:

- understand the idea of a sequence
- find the terms of a sequence
- find the formula for the general term of a sequence

10.1 NUMBER PATTERNS AND SEQUENCES

1. A **number sequence** is an ordered list of numbers generated according to a particular rule.

2. Each number in a sequence is called a **term**.

WORKED EXAMPLE 1

The following shows a sequence of numbers.
4, 6, 8, 10, 12, 14, 16, 18

(a) State the number of terms in the sequence.

(b) Identify
  (i) the first term,
  (ii) the fourth term,
  (iii) the seventh term,
  (iv) the last term in the sequence.

Worked Solution:

(a) There are **8** terms in the sequence.

(b) (i) The first term is **4**.
  (ii) The fourth term is **10**.
  (iii) The seventh term is **16**.
  (iv) The last term is **18**.

Worked Example 2

Find the next three terms for each of the following sequences.

(a) 75, 100, 125, …

(b) 7, 0, –6, …

(c) 2250, 1125, $562\frac{1}{2}$, …

(d) 64, 81, 100, …

(e) 67, 71, 73, …

Worked Solution:

(a) 75, 100, 125, **150, 175, 200**

+ 25   + 25   + 25   + 25   + 25

The next three terms are **150**, **175** and **200**.

(b) 7, 0, –6, **–11, –15, –18**

– 7   – 6   – 5   – 4   – 3

The next three terms are **–11**, **–15** and **–18**.

(c) 2250, 1125, $562\frac{1}{2}$, $\mathbf{281\frac{1}{4}}$, $\mathbf{140\frac{5}{8}}$, $\mathbf{70\frac{5}{16}}$

÷ 2   ÷ 2   ÷ 2   ÷ 2   ÷ 2

The next three terms are $\mathbf{281\frac{1}{4}}$, $\mathbf{140\frac{5}{8}}$ and $\mathbf{70\frac{5}{16}}$.

(d) 64, 81, 100, **121, 144, 169**

8^2 9^2 10^2 11^2 12^2 13^2

This sequence is made up of consecutive perfect squares. The next three terms are **121**, **144** and **169**.

(e) 67, 71, 73, **73, 71, 67**

+ 4   + 2   + 0   – 2   – 4

This sequence is made up of consecutive prime numbers. Hence, the next three terms are **73**, **71** and **67**.

WORKED EXAMPLE 3

The figure below shows the first three patterns in a sequence.

(a) Determine and draw Pattern 5.

(b) State the total number of circles in

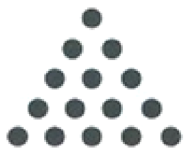
(i) Pattern 8,

(ii) Pattern 15.

Worked Solution:

Observe that there is one circle at the bottom layer in Pattern 1, two circles at the bottom layer in Pattern 2 and three circles at the bottom layer in Pattern 3.

(a) Hence, there will be five circles at the bottom layer in Pattern 5.



(b) (i) There will be eight circles at the bottom layer in Pattern 8.
Total number of circles in Pattern 8
$= 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$
$=$ **36**

(ii) There will be fifteen circles at the bottom layer in Pattern 15.
Total number of circles in Pattern 15
$= 15 + 14 + 13 + 12 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$
$=$ **120**

10.2 GENERAL TERM OF A SEQUENCE

1. The **general term** of a sequence is the nth term of the sequence.
2. Any term in a given sequence can be determined by substituting the corresponding value into the formula for the general term.

Worked Example 4

The general term of a sequence is given as $T_n = \left(\frac{2n-1}{n}\right)^2$. Find

(a) the 9th term,

(b) the difference between the 100th term and the 50th term.

Worked Solution:

(a) Substitute $n = 9$ into $T_n = \left(\frac{2n-1}{n}\right)^2$.

$$\begin{aligned} T_9 &= \left[\frac{2(9)-1}{9}\right]^2 \\ &= \left(\frac{18-1}{9}\right)^2 \\ &= \left(\frac{17}{9}\right)^2 \\ &= \frac{289}{81} \\ &= \mathbf{3\frac{46}{81}} \end{aligned}$$

(b) Substitute $n = 100$ into $T_n = \left(\frac{2n-1}{n}\right)^2$.

$$\begin{aligned} T_{100} &= \left[\frac{2(100)-1}{100}\right]^2 \\ &= \left(\frac{200-1}{100}\right)^2 \\ &= \left(\frac{199}{100}\right)^2 \\ &= \frac{39\ 601}{10\ 000} \\ &= 3.9601 \end{aligned}$$

Substitute $n = 50$ into $T_n = \left(\frac{2n-1}{n}\right)^2$.

$$T_{50} = \left[\frac{2(50)-1}{50}\right]^2$$
$$= \left(\frac{100-1}{50}\right)^2$$
$$= \left(\frac{99}{50}\right)^2$$
$$= \frac{9801}{2500}$$
$$= 3.920\,4$$

$$T_{100} - T_{50} = 3.9601 - 3.920\,4$$
$$= \mathbf{0.039\,7}$$

WORKED EXAMPLE 5

Consider the sequence $-5, -3, -1, \ldots$

(a) Find the general term of the sequence.

(b) Hence, find the 36th term.

(c) Find the value of n for the term 259.

Worked Solution:

(a) $T_1 = -5 = 2 - 7 = 2(1) - 7$
$T_2 = -3 = 4 - 7 = 2(2) - 7$
$T_3 = -1 = 6 - 7 = 2(3) - 7$
Hence, the general term is $\mathbf{2n - 7}$.

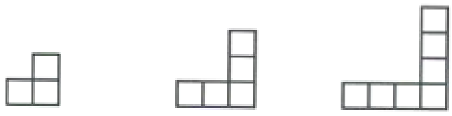
(b) $T_{36} = 2(36) - 7 = \mathbf{65}$

(c)
$$2n - 7 = 259$$
$$2n = 259 + 7$$
$$= 266$$
$$n = \frac{266}{2}$$
$$= \mathbf{133}$$

3. Steps in solving problems involving number patterns:
   (a) Understand the problem.
   (b) Look for a pattern. Draw diagrams if necessary.
   (c) Generalise the results obtained.
   (d) Check the validity of the solution.

WORKED EXAMPLE 6

The figure below shows the first three patterns in a sequence. Each of the patterns is made up of similar squares.



Pattern 1 Pattern 2 Pattern 3

(a) Study and complete the table below.

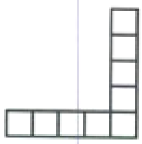
Pattern number (n)	Number of rows (R_n)	Total number of squares (T_n)
1	2	3
2	3	5
3		
4		
5		

(b) Derive a relationship between
   (i) R_n and n,
   (ii) R_n and T_n.

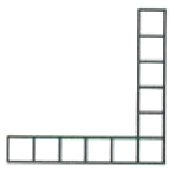
(c) Hence, find the number of rows and the total number of squares for
   (i) Pattern 23,
   (ii) Pattern 144.

Worked Solution:

(a) Pattern 4 will be as follows:



Pattern 5 will be as follows:



Pattern number (n)	Number of rows (R_n)	Total number of squares (T_n)
1	2	3
2	3	5
3	4	7
4	5	9
5	6	11

(b) (i) $R_1 = 2 = 1 + 1$
$R_2 = 3 = 2 + 1$
$R_3 = 4 = 3 + 1$
$R_4 = 5 = 4 + 1$
$R_5 = 6 = 5 + 1$
$\boldsymbol{R_n = n + 1}$

(ii) $T_1 = 3 = 2 + 1 = 2(1) + 1$
$T_2 = 5 = 4 + 1 = 2(2) + 1$
$T_3 = 7 = 6 + 1 = 2(3) + 1$
$T_4 = 9 = 8 + 1 = 2(4) + 1$
$T_5 = 11 = 10 + 1 = 2(5) + 1$
$T_n = 2n + 1$
Since $R_n = n + 1$, $n = R_n - 1$.
$T_n = 2(R_n - 1) + 1$
$= 2R_n - 2 + 1$
$\boldsymbol{T_n = 2R_n - 1}$

(c) (i) For Pattern 23:
$R_{23} = 23 + 1 = \mathbf{24}$
$T_{23} = 2R_{23} - 1$
$= 2(24) - 1$
$= 48 - 1$
$= \mathbf{47}$

(ii) For Pattern 144:
$R_{144} = 144 + 1 = \mathbf{145}$
$T_{144} = 2R_{144} - 1$
$= 2(145) - 1$
$= 290 - 1$
$= \mathbf{289}$

PRACTICE QUESTIONS

1. Find the values of p, q and r in each of the following:

(a) 60, 24, p, q, 1.536, r

(b) p, q, –56, –80, –110, –146, r

(c) 52, 55, p, 67, 78, q, r, 127, 150

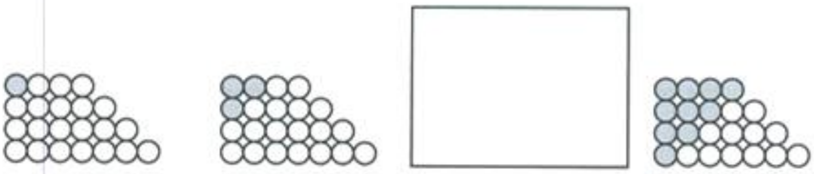
(d) 228, p, 22, q, –192, –302, r, –528

(e) 50, p, 82, 101, q, 145, 170, r, 226

(f) –27, –64, p, –216, q, –512, r, –1000

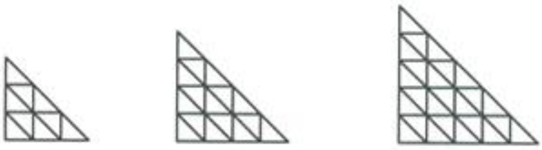
(g) p, 265, $142\frac{1}{2}$, q, r, $35\frac{5}{16}$, $27\frac{21}{32}$

2. (a) The figure below shows the first four patterns in a sequence. Each pattern is made up of two different types of circles. Draw Pattern 3.



Pattern 1 Pattern 2 Pattern 3 Pattern 4

(b) The figure below shows the first three patterns in a sequence. Each pattern is made up of triangles of equal size.



Pattern 1 Pattern 2 Pattern 3

(i) Draw Pattern 4.

(ii) Find the total number of triangles in Pattern 11.

3. Observe the following sequence:
Line 1: $1^2 = 1$
Line 2: $1^2 + 3^2 = 10$
Line 3: $1^2 + 3^2 + 5^2 = 35$
(a) Write down the pattern for Line 14.
(b) Find the difference between the sums of Line 7 and Line 14.

4. The general term of a sequence is given by $T_n = \frac{3}{4}\left[n^2 + \left(\frac{n+7}{2}\right)\right]^3$. Find
(a) T_1,
(b) T_5,
(c) T_{16}.

5. (a) Given the sequence –24, –48, –72, –96, …, find
(i) the nth term,
(ii) the 75th term,
(iii) the term in the sequence that gives –3312.
(b) Given the sequence 120, 132, 144, 156, …, find
(i) the nth term,
(ii) the 33rd term,
(iii) the value of p if the pth term in the sequence is 1308.

6. The following shows a sequence.

$$-6,\ 1^3 - 36,\ 2^3 - 216,\ 3^3 - 1296,\ \ldots$$

(a) Write the next term in the same form.
(b) Find the general term, in terms of n, of the given sequence.
(c) Hence, write the 16th term of the sequence in the same form.

7. The following shows a pattern.
$2^2 - 1^2 = 3$
$3^2 - 2^2 = 5$
$4^2 - 3^2 = 7$
$5^2 - 4^2 = 9$

(a) Complete the table below.

Line number, n	Pattern, P_n	Sum, S_n
1	$2^2 - 1^2$	3
2		
3		
4		

(b) Find the general term, in terms of n, for P_n and S_n.
(c) Hence, find
(i) P_{85}
(ii) S_{150}.
(d) Use the pattern to find the value of $180^2 - 179^2$.