CHAPTER 3 Expansion and Factorisation of Algebraic Expressions

3.1 Expansion of Algebraic Expressions

3.1.1 Removing Brackets

1. Recall the following properties which apply to algebraic operations.

   - Distributive property of multiplication over addition.

     Example: $a(b + c) = ab + ac$

   - Distributive property of multiplication over subtraction.

     Example: $a(b - c) = ab - ac$

   - $(-) \times (+) = (-)$
     $(-) \times (-) = (+)$

     Examples: (a) $-a(b + c) = -ab - ac$

     (b) $-a(b - c) = -ab + ac$

2. When an expression within brackets is multiplied by a number, each term within the brackets must be multiplied by the same number when the brackets are removed. Any change in operation signs must be observed.

   Examples: (a) $a(b + 3) = ab + 3a$

   (b) $-3a(b - 1) = -3ab + 3a$

   (c) $4a(c - 5d) + a(c + d) = 4ac - 20ad + ac + ad$
   $= 5ac - 19ad$

   (d) $3a(b + c) - 4a(3b - 2c) = 3ab + 3ac - 12ab + 8ac$
   $= 11ac - 9ab$

Practice 3.1.1

Basic

1. Expand and simplify the following expressions.

(a) $2a(c + 3b)$
(b) $-5c(3a - 5b)$
(c) $2e(2f + g) - e(f - g)$
(d) $h(2g - k) - 2h(g + 2k)$
(e) $2a(3b - 2c) + a(c - b)$
(f) $3d(2e - f) - 2f(d - e)$
(g) $\frac{1}{2}a(2b - 4c) + \frac{3}{4}a(4b - 8c)$
(h) $\frac{3}{8}x(24y - 16z) - \frac{3}{5}x(5z - 15y)$

3.1.2 Expansion Using Special Algebraic Identities

1. Note the following expansion of algebraic expressions which involve special algebraic identities.

Examples: (a)
$$\begin{aligned}\boldsymbol{(a + b)^2} &= (a + b)(a + b)\\ &= a(a + b) + b(a + b)\\ &= a^2 + ab + ab + b^2\\ &= \boldsymbol{a^2 + 2ab + b^2}\end{aligned}$$

(b)
$$\begin{aligned}\boldsymbol{(a - b)^2} &= (a - b)(a - b)\\ &= a(a - b) - b(a - b)\\ &= a^2 - ab - ab + b^2\\ &= \boldsymbol{a^2 - 2ab + b^2}\end{aligned}$$

(c)
$$\begin{aligned}\boldsymbol{(a + b)(a - b)} &= a(a - b) + b(a - b)\\ &= a^2 - ab + ab - b^2\\ &= \boldsymbol{a^2 - b^2}\end{aligned}$$

2. The result of $\boldsymbol{(a + b)^2 = a^2 + 2ab + b^2}$ can be used to expand $(3a + 2b)^2$.
$$\begin{aligned}(3a + 2b)^2 &= (3a)^2 + 2(3a)(2b) + (2b)^2\\ &= 9a^2 + 12ab + 4b^2\end{aligned}$$

3. The result of $\boldsymbol{(a - b)^2 = a^2 - 2ab + b^2}$ can be used to expand $(5b - 3c)^2$.
$$\begin{aligned}(5b - 3c)^2 &= (5b)^2 - 2(5b)(3c) + (3c)^2\\ &= 25b^2 - 30bc + 9c^2\end{aligned}$$

4. The result of $\boldsymbol{(a + b)(a - b) = a^2 - b^2}$ can be used to expand $(7m + 2n)(7m - 2n)$.
$$\begin{aligned}(7m + 2n)(7m - 2n) &= (7m)^2 - (2n)^2\\ &= 49m^2 - 4n^2\end{aligned}$$

Practice 3.1.2

Basic

1. Expand the following expressions.

(a) $(a+5)^2$

(b) $(b-8)^2$

(c) $(3c+4)^2$

(d) $(2d-5)^2$

(e) $(4e+3f)^2$

(f) $(7f-2g)^2$

(g) $\left(g+\frac{2}{3}\right)^2$

(h) $\left(h-\frac{3}{4}\right)^2$

(i) $\left(\frac{1}{4}k+\frac{1}{2}m\right)^2$

(j) $\left(\frac{2}{3}p-\frac{1}{5}q\right)^2$

(k) $(2k+3)(2k-3)$

(l) $(5m+3n)(5m-3n)$

(m) $\left(\frac{4}{5}p+q\right)\left(\frac{4}{5}p-q\right)$

(n) $\left(-3r+\frac{2}{3}s\right)\left(-3r-\frac{2}{3}s\right)$

2. Expand the following expressions.

(a) $(a+bc)^2$

(b) $(b-cd)^2$

(c) $(3cd+2)^2$

(d) $(4de-3)^2$

(e) $(1+ef)^2$

(f) $(3-2fg)^2$

(g) $\left(g+\frac{1}{4}h\right)^2$

(h) $\left(\frac{3}{4}e-gh\right)^2$

(i) $\left(\frac{3}{5}ac+\frac{1}{2}b\right)^2$

(j) $\left(\frac{2}{5}c-\frac{1}{3}ad\right)^2$

(k) $(3ef+d)(3ef-d)$

(l) $(4fg+5)(4fg-5)$

(m) $\left(\frac{2}{3}hk+4n\right)\left(\frac{2}{3}hk-4n\right)$

(n) $\left(-2mn-\frac{3}{5}p\right)\left(-2mn+\frac{3}{5}p\right)$

3.1.3 Further Expansions of Other Algebraic Expressions

Examples: (a) Expand and simplify $(a + b)(a + b + c)$.

$$\begin{aligned}(a + b)(a + b + c) &= a(a + b + c) + b(a + b + c)\\ &= a^2 + ab + ac + ab + b^2 + bc\\ &= a^2 + 2ab + ac + bc + b^2\end{aligned}$$

(b) Expand and simplify $(p + 4)(p - 5) - (p - 1)(p + 3)$.

$$\begin{aligned}(p + 4)(p - 5) - (p - 1)(p + 3) &= p^2 - 5p + 4p - 20 - (p^2 + 3p - p - 3)\\ &= p^2 - 5p + 4p - 20 - p^2 - 3p + p + 3\\ &= -3p - 17\end{aligned}$$

Practice 3.1.3

Basic

1. Expand and simplify the following expressions.

(a) $(a + b)(a - b - c)$
(b) $(a + 2)(a + 3)(a + 4)$
(c) $(m + 2)^3$
(d) $(m - n + 2)^2$
(e) $(x - 3)(x^2 - 2x + 5)$
(f) $(p + q)^2 - (p - q)^2$
(g) $(q + 2)(q - 3) + (2q + 1)(q - 2)$
(h) $(3a + b)(a - b) - (2a + 3b)(a - 2b)$

2. Expand and simplify the following expressions.

(a) $(c + b)(c - b + d)$
(b) $(e + f)(e - f)^2$
(c) $(1 + h)(2 + h)(3 + h)$
(d) $(2 - k)(3 - k)(4 + k)$
(e) $m(m + 4)(m^2 - 2m)$
(f) $(p + 2)^2 - (p - 3)^2$
(g) $(q + 2)(q + 3) - (2q + 1)(3q - 2)$

3.2 Factorisation of Algebraic Expressions

Factorisation of an algebraic expression is the process of writing the expression as a product of its factors.

3.2.1 Factorisation by Using Common Factors

Examples: (a) In the expression $4x + 6y - 8z$, the common factor to all three terms is 2.
Thus, $4x + 6y - 8z = 2(2x + 3y - 4z)$.

(b) $15a + 20b - 10c = 5(3a + 4b - 2c)$

(c) $bc^2 - b^2c = bc(c - b)$

(d) $15x^2 + 5x = 5x(3x + 1)$

(e) $x(a + b) - y(a + b) = (a + b)(x - y)$

Practice 3.2.1

Basic

1. Factorise the following expressions.

(a) $4a + 12bc$

(b) $20bc - 5cd$

(c) $32cd + 12de$

(d) $5de + 10d^2e$

(e) $15e^2f - 25ef^2 + 5ef$

(f) $10abc + 15a^2bc - 5abc^2$

(g) $2\pi rh + 2\pi r^2 - 2\pi rl$

2. Factorise the following expressions.

(a) $5ac - 25bc$

(b) $12bd + 6cd - 3d$

(c) $3c^2 - 6ce + 9c^2e$

(d) $25bcd + 15b^3cd^2 - 5bc$

(e) $18ef^2 - 9e^2f + 27ef$

(f) $6fgh + 12f^2gh - 18fgh^2$

(g) $4(a + 3b) + 3c(a + 3b)$

(h) $3c(2d - e) - 4(2d - e)$

Advanced

3. Factorise the following.

(a) $d(a - c) - e(c - a)$

(b) $3(e - f) - a(f - e)$

3.2.2 Factorisation by Grouping

In the expression $ab + cd + ad + cd$, the first two terms have the common factor b and the last two terms have the common factor d. By grouping the first two terms and factorising them, we obtain $b(a + c)$. By grouping the last two terms and factorising them, we obtain $d(a + c)$.

Thus, $ab + cd + ad + cd = b(a + c) + d(a + c)$
$= (a + c)(b + d)$

Examples: (a) Factorise $ax - ay + bx - by$.

$$\begin{aligned} ax - ay + bx - by &= a(x - y) + b(x - y) \\ &= (x - y)(a + b) \end{aligned}$$

(b) Factorise $ad + bc - ac - bd$.

$$\begin{aligned} ad + bc - ac - bd &= ad - ac + bc - bd \\ &= a(d - c) + b(c - d) \\ &= a(d - c) - b(d - c) \\ &= (d - c)(a - b) \end{aligned}$$

Practice 3.2.2

Basic

1. Factorise the following expressions.

(a) $pq + qr + ps + rs$

(b) $2ab - 3b + 2ac - 3c$

(c) $3p + pq + qr + 3r$

(d) $ac - ad - ce + de$

(e) $4am - 4an + 5n - 5m$

(f) $2qx - 3px + 6py - 4qy$

(g) $2rs - 4ru - 10pu + 5ps$

(h) $6ad - 9aq + 3q - 2d$

2. Factorise the following expressions.

(a) $ah + bh + ak + bk$

(b) $2bc - 3b + 2ac - 3a$

(c) $4c + ce + em + 4m$

(d) $cd - de - cf + ef$

(e) $6em - 6en + 5n - 5m$

(f) $2fg - 3gh + 6hp - 4fp$

(g) $2gh - 4gk - 10jk + 5hj$

(h) $15hq - 10hp + 2p - 3q$

Advanced

3. Factorise the following expressions.

(a) $(a + b)^2 + (a + b)$

(b) $(a - c) - (a - c)^2$

3.2.3 Factorisation of Quadratic Expressions

1. An expression of $ax^2 + bx + c$, where a, b and c are real numbers and $a \neq 0$, is called a quadratic expression.

Examples: (a) $x^2 + x$ is a quadratic expression where $a = 1$, $b = 1$ and $c = 0$.

(b) $3x^2 + x - 2$ is a quadratic expression where $a = 3$, $b = 1$ and $c = -2$.

(c) $5x^2 - 2x - 3$ is a quadratic expression where $a = 5$, $b = -2$ and $c = -3$.

2. When a quadratic expression takes the pattern of perfect squares, its factors may be expressed in the form $(\boldsymbol{a} + \boldsymbol{b})^2$ or $(\boldsymbol{a} - \boldsymbol{b})^2$.

Examples: (a) Factorise $a^2 + 4a + 4$.

$$\begin{aligned} a^2 + 4a + 4 &= a^2 + 2(a)(2) + 2^2 \\ &= (a + 2)^2 \end{aligned}$$

Recall that $a^2 + 2ab + b^2 = (a + b)^2$.

(b) Factorise $9x^2 - 24x + 16$.

$$\begin{aligned} 9x^2 - 24x + 16 &= (3x)^2 - 2(3x)(4) + 4^2 \\ &= (3x - 4)^2 \end{aligned}$$

Recall that $a^2 - 2ab + b^2 = (a - b)^2$.

3. When a quadratic expression takes the pattern of the difference of two squares, its factors may be expressed in the form $(\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{a} - \boldsymbol{b})$.

Examples: (a) Factorise $c^2 - 9d^2$.

$$\begin{aligned} c^2 - 9d^2 &= c^2 - (3d)^2 \\ &= (c + 3d)(c - 3d) \end{aligned}$$

Recall that $a^2 - b^2 = (a + b)(a - \text{b})$.

(b) Factorise $25m^2 - 36n^2$.

$$\begin{aligned} 25m^2 - 36n^2 &= (5m)^2 - (6n)^2 \\ &= (5m + 6n)(5m - 6n) \end{aligned}$$

4. We can use a **multiplication frame** to help us factorise a quadratic equation.

Examples: (a) Factorise $c^2 + 9c + 18$.

Step 1:

$$\begin{array}{cc|c} & & \\ \hline c^2 & 18 & \end{array}$$

Step 2:

$$\begin{array}{cc|c} c & 3 & \\ c & 6 & \\ \hline c^2 & 18 & \end{array}$$

Step 3:

$$\begin{array}{cc|c} c & 3 & 3c \\ c & 6 & 6c \\ \hline c^2 & 18 & 9c \end{array}$$

Check: $3c + 6c = 9c$

$c^2 + 9c + 18 = (c + 3)(c + 6)$

(b) Factorise $6c^2 - 7c - 20$.

Step 1:

$$\begin{array}{cc|c} & & \\ \hline 6c^2 & -20 & \end{array}$$

Step 2:

$$\begin{array}{cc|c} 3c & 4 & \\ 2c & -5 & \\ \hline 6c^2 & -20 & \end{array}$$

Step 3:

$$\begin{array}{cc|c} 3c & 4 & 8c \\ 2c & -5 & -15c \\ \hline 6c^2 & -20 & -7c \end{array}$$

Check: $8c - 15c = -7c$

$6c^2 + 7c - 20 = (3c + 4)(2c - 5)$

Practice 3.2.3

Basic

1. Factorise the following expressions.

(a) $a^2 + 6a + 9$
(b) $c^2 - 10c + 25$
(c) $4p^2 + 12pq + 9q^2$
(d) $36e^2 - 12ef + f^2$
(e) $a^2 + 8a + 16$
(f) $b^2 - 14b + 49$
(g) $9c^2 + 12cd + 4d^2$
(h) $16d^2 - 8de + e^2$

2. Factorise the following expressions.

(a) $4s^2 - t^2$
(b) $\frac{1}{4}m^2 - 64$
(c) $81p^2q^2 - 121r^2s^2$
(d) $(a + b)^2 - (a - b)^2$
(e) $e^2 - 25f^2$
(f) $64f^2 - \frac{1}{9}h^2$
(g) $49g^2h^2 - 100k^2$
(h) $(h - 2j)^2 - k^4$

3. Factorise the following expressions.

(a) $f^2 + 5f + 6$
(b) $4r^2 - 5r + 1$
(c) $6x^2 + 13x - 5$
(d) $2m^2 - 7mn - 4n^2$
(e) $9n^2 + 8n - 1$
(f) $4p^2 - 7pq - 2q^2$

Advanced

4. Factorise the following expressions.

(a) $(a + b)^2 + 5(a + b) + 6$
(b) $3 - 7(c - d) + 2(c - d)^2$