5 EXPANSION AND FACTORISATION USING SPECIAL ALGEBRAIC IDENTITIES

LEARNING OBJECTIVES

In this topic, we will learn to:

- apply three special algebraic identities to expand algebraic expressions
- apply three special algebraic identities to factorise algebraic expressions

5.1 EXPANSION USING SPECIAL ALGEBRAIC IDENTITIES

1. Algebraic expressions of the form of perfect squares can be expanded as
   (a) $(a+b)^2 = a^2 + 2ab + b^2$.
   (b) $(a-b)^2 = a^2 - 2ab + b^2$.

> *Note:*
> $(a+b)^2 = (a+b)(a+b) = a^2 + ab + ba + b^2$
> $(a-b)^2 = (a-b)(a-b) = a^2 - ab - ba + b^2$

2. Algebraic expressions of the form of a difference of two squares can be expanded as $(a+b)(a-b) = a^2 - b^2$.

WORKED EXAMPLE 1

Expand the following.

(a) $(x+3)^2$ (b) $(5x+1)^2$ (c) $(2y+7x)^2$

Worked Solution:

(a) $(x+3)^2 = x^2 + 2(x)(3) + 3^2$
$\quad = \mathbf{x^2 + 6x + 9}$

(b) $(5x+1)^2 = (5x)^2 + 2(5x)(1) + 1^2$
$\quad = \mathbf{25x^2 + 10x + 1}$

(c) $(2y+7x)^2 = (2y)^2 + 2(2y)(7x) + (7x)^2$
$\quad = \mathbf{4y^2 + 28xy + 49x^2}$

> ***Student's common mistake:***
> In part (b) of the example, writing $(5x)^2$ as $5x^2$ is wrong. $(5x)^2 = 5x \times 5x = 25x^2$.

Worked Example 2

Expand the following.

(a) $(x-3)^2$ (b) $(3-2x)^2$ (c) $(2y-7x)^2$

Worked Solution:

(a) $(x-3)^2 = x^2 - 2(x)(3) + 3^2$
$= \mathbf{x^2 - 6x + 9}$

(b) $(3-2x)^2 = 3^2 - 2(3)(2x) + (2x)^2$
$= \mathbf{9 - 12x + 4x^2}$

(c) $(2y-7x)^2 = (2y)^2 - 2(2y)(7x) + (7x)^2$
$= \mathbf{4y^2 - 28xy + 49x^2}$

Worked Example 3

Expand the following.

(a) $(x+7)(x-7)$ (b) $(2x-5y)(2x+5y)$

Worked Solution:

(a) $(x+7)(x-7) = x^2 - 7^2$
$= \mathbf{x^2 - 49}$

(b) $(2x-5y)(2x+5y) = (2x)^2 - (5y)^2$
$= \mathbf{4x^2 - 25y^2}$

Worked Example 4

Given that $m^2 + n^2 = 14$ and $mn = 6$, find the value of $(m+n)^2$.

Worked Solution:

$(m+n)^2 = m^2 + 2mn + n^2$
$= m^2 + n^2 + 2mn$
$= 14 + 2(6)$
$= \mathbf{26}$

WORKED EXAMPLE 5

Evaluate the following without using a calculator.

(a) 105^2 (b) 399^2 (c) 128×132

Worked Solution:

(a) $105^2 = (100 + 5)^2$
$= 100^2 + 2(100)(5) + 5^2$
$= 10\,000 + 1000 + 25$
$= \mathbf{11\,025}$

(b) $399^2 = (400 - 1)^2$
$= 400^2 - 2(400)(1) + 1^2$
$= 160\,000 - 800 + 1$
$= \mathbf{159\,201}$

(c) $128 \times 132 = (130 - 2)(130 + 2)$
$= 130^2 - 2^2$
$= 16\,900 - 4$
$= \mathbf{16\,896}$

5.2 FACTORISATION USING SPECIAL ALGEBRAIC IDENTITIES

1. Factorisation is the opposite of expansion.
   (a) $a^2 + 2ab + b^2 = (a + b)^2$
   (b) $a^2 - 2ab + b^2 = (a - b)^2$
   (c) $a^2 - b^2 = (a + b)(a - b)$

WORKED EXAMPLE 6

Factorise the following completely.

(a) $x^2 + 4x + 4$
(b) $9x^2 + 6x + 1$
(c) $4y^2 + 20xy + 25x^2$

Worked Solution:

(a) $x^2 + 4x + 4 = x^2 + 2(x)(2) + 2^2$
$= \mathbf{(x + 2)^2}$

(b) $9x^2 + 6x + 1 = (3x)^2 + 2(3x)(1) + 1^2$
$= \mathbf{(3x + 1)^2}$

(c) $4y^2 + 20xy + 25x^2 = (2y)^2 + 2(2y)(5x) + (5x)^2$
$= \mathbf{(2y + 5x)^2}$

Worked Example 7

Factorise the following completely.

(a) $x^2 - 8x + 16$

(b) $8x^2 - 40x + 50$

(c) $9x^2 - 12xy + 4y^2$

Worked Solution:

(a) $x^2 - 8x + 16 = x^2 - 2(x)(4) + 4^2$
$= \mathbf{(x - 4)^2}$

(b) $8x^2 - 40x + 50 = 2(4x^2 - 20x + 25)$
$= 2[(2x)^2 - 2(2x)(5) + 5^2]$
$= \mathbf{2(2x - 5)^2}$

(c) $9x^2 - 12xy + 4y^2 = (3x)^2 - 2(3x)(2y) + (2y)^2$
$= \mathbf{(3x - 2y)^2}$

Worked Example 8

Factorise the following completely.

(a) $4x^2 - 49$

(b) $12y^2 - 3x^2$

Worked Solution:

(a) $4x^2 - 49 = (2x)^2 - 7^2$
$= \mathbf{(2x + 7)(2x - 7)}$

(b) $12y^2 - 3x^2 = 3(4y^2 - x^2)$
$= 3[(2y)^2 - x^2]$
$= \mathbf{3(2y + x)(2y - x)}$

WORKED EXAMPLE 9

Evaluate the following without using a calculator.

(a) $90^2 - 10^2$ (b) $102^2 - 4$

Worked Solution:

(a) $90^2 - 10^2 = (90 + 10)(90 - 10)$
$= 100 \times 80$
$= \mathbf{8000}$

(b) $102^2 - 4 = 102^2 - 2^2$
$= (102 + 2)(102 - 2)$
$= 104 \times 100$
$= \mathbf{10\ 400}$

PRACTICE QUESTIONS

1. Expand the following.
   (a) $(x + 5)^2$ (b) $(3x + 1)^2$
   (c) $(2 + 3x)^2$ (d) $(7x + 9y)^2$

2. Expand the following.
   (a) $(x - 4)^2$ (b) $(2x - 5)^2$
   (c) $(6 - x)^2$ (d) $(x - 3y)^2$

3. Expand the following.
   (a) $(x + 5)(x - 5)$ (b) $(3x - 5y)(3x + 5y)$

4. Given that $x^2 + y^2 = 1000$ and $xy = 56$, find the value of $(x + y)^2$.

5. Given that $m^2 - n^2 = 48$ and $m - n = 5$, find the value of $2(m + n)^2$.

6. Evaluate the following without using a calculator.
   (a) 102^2 (b) 48^2 (c) 196×204

7. Factorise the following completely.
   (a) $x^2 + 6x + 9$
   (b) $9x^2 + 12x + 4$
   (c) $25x^2 + 30xy + 9y^2$

8. Factorise the following completely.
   (a) $x^2 - 6x + 9$
   (b) $12x^2 - 12x + 3$
   (c) $4x^2 - 12xy + 9y^2$

9. Factorise the following completely.
   (a) $4x^2 - 9$ (b) $x^2 - 25y^2$

10. Evaluate the following without using a calculator.
   (a) $75^2 - 25^2$ (b) $105^2 - 25$

11. It is given that x is a positive integer.
   (a) Stella thinks that $2x + 3$ is an odd number. Is she correct? Explain your answer.
   (b) Find an expression for the square of the closest odd number which is smaller than $2x + 3$.