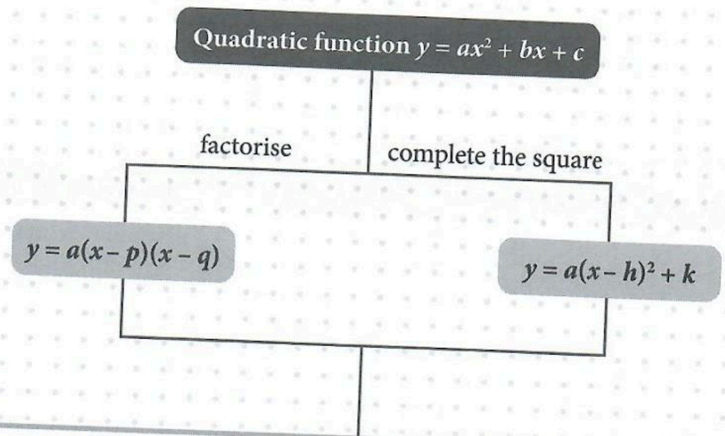
CHAPTER 1 Quadratic Functions

Chapter Summary



To sketch a quadratic graph $y = ax^2 + bx + c$:

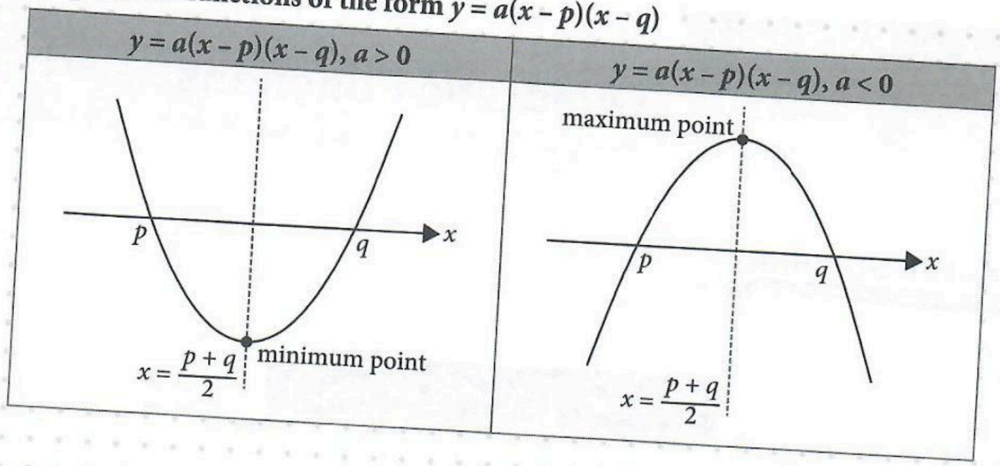
Step 1: Determine whether the graph is ∪-shaped or ∩-shaped.

Step 2: Express the function as:

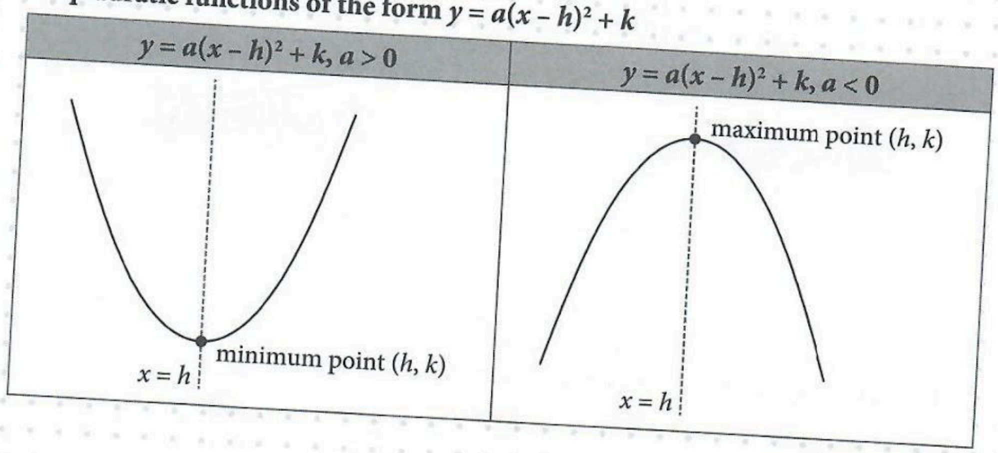
- $y = a(x - p)(x - q)$ to obtain the x-intercepts, or
- $y = a(x - h)^2 + k$ to obtain the coordinates of the turning point.

Step 3: Substitute $x = 0$ into the equation of the curve to find the y-intercept.

Graphs of quadratic functions of the form $y = a(x - p)(x - q)$



Graphs of quadratic functions of the form $y = a(x - h)^2 + k$



1.1 Completing the square

Objectives Checklist

- Complete the square
- Find the maximum or minimum value of a quadratic function by completing the square

Notes and Worked Examples

A quadratic expression is of the form $ax^2 + bx + c$, where a, b and c are real numbers and $a \neq 0$.

The process of expressing $x^2 + px$ in the form $(x - h)^2 + k$ is known as **completing the square.**

To complete the square of a quadratic expression $x^2 + px$, we add $\left(\frac{p}{2}\right)^2$ to the expression:

$$x^2 + px = \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2$$

WORKED EXAMPLE 1 Completing the square

Express each of the following in the form $a(x - h)^2 + k$.

(a) $x^2 + 8x + 1$

(b) $3x^2 - 6x - 2$

(c) $-2x^2 - 10x + 5$

(d) $\frac{1}{7}x^2 + x - 4$

SOLUTION

(a)
$$\begin{aligned} x^2 + 8x + 1 &= x^2 + 8x + 4^2 - 4^2 + 1 \\ &= (x + 4)^2 - 4^2 + 1 \\ &= \mathbf{(x + 4)^2 - 15} \end{aligned}$$

(b)
$$\begin{aligned} 3x^2 - 6x - 2 &= 3(x^2 - 2x) - 2 \\ &= 3[x^2 - 2x + (-1)^2 - (-1)^2] - 2 \\ &= 3[(x - 1)^2 - 1^2] - 2 \\ &= 3(x - 1)^2 - 3 - 2 \\ &= \mathbf{3(x - 1)^2 - 5} \end{aligned}$$

When the coefficient of $x^2 \neq 1$, we factorise the expression before completing the square.

(c) $-2x^2 - 10x + 5 = -2(x^2 + 5x) + 5$

$$= -2\left[x^2+5x+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2\right]+5$$

$$= -2\left[\left(x+\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2\right]+5$$

$$= -2\left(x+\frac{5}{2}\right)^2+\frac{25}{2}+5$$

$$= \mathbf{-2\left(x+\frac{5}{2}\right)^2+\frac{35}{2}}$$

(d) $\frac{1}{7}x^2+x-4 = \frac{1}{7}(x^2+7x)-4$

$$= \frac{1}{7}\left[x^2+7x+\left(\frac{7}{2}\right)^2-\left(\frac{7}{2}\right)^2\right]-4$$

$$= \frac{1}{7}\left[\left(x+\frac{7}{2}\right)^2-\left(\frac{7}{2}\right)^2\right]-4$$

$$= \frac{1}{7}\left(x+\frac{7}{2}\right)^2-\frac{7}{4}-4$$

$$= \mathbf{\frac{1}{7}\left(x+\frac{7}{2}\right)^2-\frac{23}{4}}$$

TRY TUTORIAL 1.1: Question 1

WORKED EXAMPLE 2 Finding the maximum or minimum value of a quadratic function

(i) Show that $x^2 - 6x + 11 \geqslant 2$ for all real values of x.

(ii) State the maximum or minimum value of $x^2 - 6x + 11$ and the corresponding value of x.

SOLUTION

(i) $x^2 - 6x + 11 = x^2 - 6x + (-3)^2 - (-3)^2 + 11$

$$= (x-3)^2 + 2$$

Since $(x-3)^2 \geqslant 0$ for all real values of x, then $(x-3)^2 + 2 \geqslant 2$.

$\therefore$ $x^2 - 6x + 11 \geqslant 2$ for all real values of x. (shown)

(ii) **Minimum** value of $x^2 - 6x + 11$ is **2**, when $x = \mathbf{3}$

TRY TUTORIAL 1.1: Questions 2–5

NAME: ____________________ CLASS: ____________ DATE: __________

TUTORIAL **1.1**

1. Express each of the following in the form $a(x - h)^2 + k$.

   (a) $x^2 - 4x$ (b) $2x^2 + 12x + 3$

   (c) $-x^2 + 8x - 7$ (d) $-\frac{1}{2}x^2 - 5x + 6$

2. State the maximum or minimum value of each of the following and the corresponding value of x.

   (a) $(x - 3)^2 + 5$ (b) $-x^2 + 9$

   (c) $2(x+4)^2 - \frac{7}{10}$ (d) $-\frac{1}{6}(x-1)^2 - 8$

3. Find the maximum or minimum value of each of the following and the corresponding value of x.

   (a) $2x^2 - 8x + 9$ (b) $-x^2 - x + 1$

   (c) $\frac{4}{3}x^2 - 12x - 5$ (d) $7 - 6x - 0.2x^2$

4. Show that $x^2 + 4x - 3 \geqslant -7$ for all real values of x.

5. Explain why $x - 5x^2 - 1$ is always negative for all real values of x.

1.2 Quadratic functions of the form $y = a(x - p)(x - q)$ and $y = a(x - h)^2 + k$

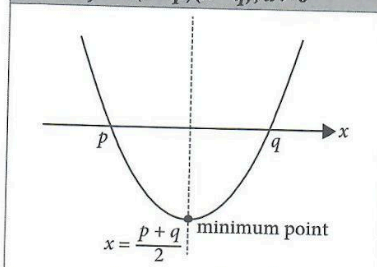
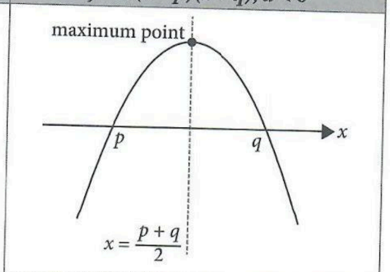
Objectives Checklist

- Sketch the graph of a quadratic function
- Find the maximum or minimum value of a quadratic function by completing the square
- State the conditions for a quadratic curve to lie completely above or below the x-axis

Notes and Worked Examples

The three **equivalent** forms of quadratic functions are $y = ax^2 + bx + c$, $y = a(x - p)(x - q)$ and $y = a(x - h)^2 + k$, where $a \neq 0$.

Graphs of quadratic functions of the form $y = a(x - p)(x - q)$

$y = a(x - p)(x - q)$, $a > 0$	$y = a(x - p)(x - q)$, $a < 0$
	

Graphs of quadratic functions of the form $y = a(x - h)^2 + k$

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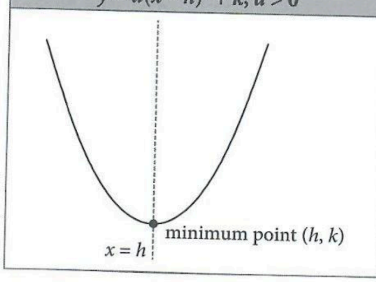	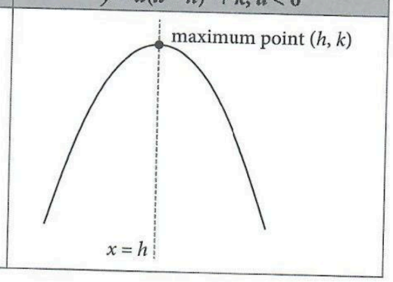

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To sketch a quadratic graph $y = ax^2 + bx + c$:

Step 1: Determine whether the graph is ∪-shaped or ∩-shaped.

Step 2: Express the function as:

- $y = a(x - p)(x - q)$ to obtain the x-intercepts, or
- $y = a(x - h)^2 + k$ to obtain the coordinates of the turning point.

Step 3: Substitute $x = 0$ into the equation of the curve to find the y-intercept.

A quadratic curve can intersect the x-axis, or lie completely above or below the x-axis:

- $a > 0$ and minimum value of $ax^2 + bx + c > 0 \Leftrightarrow$ the curve lies completely above the x-axis
- $a < 0$ and maximum value of $ax^2 + bx + c < 0 \Leftrightarrow$ the curve lies completely below the x-axis

WORKED EXAMPLE 1 **Finding the coordinates of the turning point of a quadratic curve**

(i) Find the coordinates of the turning point of the graph of $y = -2x^2 + 8x - 3$ and sketch the graph.

(ii) State the maximum or minimum value of $y = -2x^2 + 8x - 3$.

SOLUTION

(i) $y = -2x^2 + 8x - 3$

$= -2(x^2 - 4x) - 3$

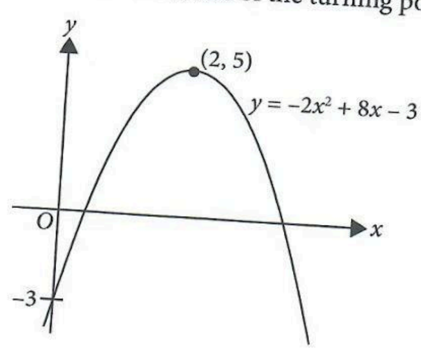
$= -2[x^2 - 4x + (-2)^2 - (-2)^2] - 3$

$= -2[(x - 2)^2 - 4] - 3$

$= -2(x - 2)^2 + 8 - 3$

$= -2(x - 2)^2 + 5$

$\therefore$ The coordinates of the turning point are **(2, 5)**.



(ii) The **maximum** value is **5**.

TRY TUTORIAL 1.2: Questions 1–5

WORKED EXAMPLE 2

Finding the conditions for the maximum value of a quadratic function to be negative

The maximum value of $hx^2 - 12x + k$ is negative.

(i) What conditions must apply to the constants h and k?

(ii) Give an example of the values of h and k which satisfy the conditions found in part (i).

SOLUTION

(i) $$hx^2 - 12x + k = h\left(x^2 - \frac{12}{h}x\right) + k$$

$$= h\left[x^2 - \frac{12}{h}x + \left(-\frac{6}{h}\right)^2 - \left(-\frac{6}{h}\right)^2\right] + k$$ ⟵ Complete the square.

$$= h\left[\left(x - \frac{6}{h}\right)^2 - \frac{36}{h^2}\right] + k$$

$$= h\left(x - \frac{6}{h}\right)^2 + k - \frac{36}{h}$$

∴ The conditions are $\boldsymbol{h < 0}$ and $\boldsymbol{k - \frac{36}{h} < 0}$. ⟵ The coefficient of x^2 must be negative for the expression to have a maximum value.

(ii) A possible set of values is $\boldsymbol{h = -1}$ and $\boldsymbol{k = -40}$.

TRY TUTORIAL 1.2: Questions 6–10

NAME: ______________________ CLASS: __________ DATE: ________

TUTORIAL 1.2

1. (i) Find the coordinates of the turning point of the graph of $y = 3x^2 - 6x + 1$ and sketch the graph.
   (ii) State the maximum or minimum value of $y = 3x^2 - 6x + 1$.

2. (i) Express $-\frac{1}{2}(2x+5)(x-1)$ in the form $a(x - h)^2 + k$.
   (ii) Hence, write down the coordinates of the turning point and sketch the graph of $y = -\frac{1}{2}(2x+5)(x-1)$.

3. Find the coordinates of the maximum or minimum point of each of the following graphs.
   (a) $y = 2x^2 - 10x$
   (b) $y = -1.6(x + 2)(x + 4)$

4. Find the minimum value of $5 - x + 3x^2$ and the value of x for which the minimum occurs.

5. The coordinates of the turning point of the graph of $y = -0.4(x + p)^2 + 2q$ are $(-1, 5)$.
   (i) Is the turning point a maximum or a minimum point? How can you tell?
   (ii) State the value of p and of q.

6. Determine whether each of the following curves lies completely above or below the x-axis.
   (a) $y = 4x^2 - 12x + 15$
   (b) $y = -\frac{1}{2}x^2 + x - 8$

7. The minimum value of $px^2 - 7x - q$ is positive.
   (i) What conditions must apply to the constants p and q?
   (ii) Give an example of the values of p and q which satisfy the conditions found in part (i).

8. The expression $k - \frac{3}{4}(x^2 + x)$, where k is a constant, can never take a value greater than -1. Frank says that the value of k is -1. Do you agree with him? Show working to support your answer.

9. The line of symmetry of a quadratic curve is $x = 3$. Write down a possible equation of the curve for which
   (a) it lies completely above the x-axis and the point $(2, 10)$ lies on the curve,
   (b) it lies completely below the x-axis and the point $(1, -7)$ lies on the curve.

10. Given the expression $5(x-2)^2+4$, which of the following statements are true? Explain yo answer.

(a) The maximum value of the expression is 5.

(b) The minimum value of the expression is 4.

(c) The graph of $y=5(x-2)^2+4$ has a minimum point at $(-2, 4)$.

(d) The graph of $y=5(x-2)^2+4$ lies completely above the x-axis only for $x \geqslant 2$.

1.3 Quadratic functions in real-world contexts

Objectives Checklist

- Use quadratic functions to model real-world objects and phenomena and solve related problems

Notes and Worked Examples

Quadratic functions can be used to **model** real-world scenarios such as projectile motion and shapes of design structures.

WORKED EXAMPLE 1 Applying quadratic functions in real-world contexts

A ball is released from a machine. Its height, y metres, above the ground can be modelled by the equation $y = -0.5x^2 + 4x + 1.2$, where x is the horizontal distance from the machine in metres.

(i) Find the greatest height reached by the ball and the corresponding horizontal distance travelled.

(ii) A small balloon is 6 m horizontally from the machine and 7.2 m above the ground. Determine whether the ball will hit the balloon.

SOLUTION

(i) $y = -0.5x^2 + 4x + 1.2 \quad — (1)$

$= -0.5(x^2 - 8x) + 1.2$

$= -0.5[x^2 - 8x + (-4)^2 - (-4)^2] + 1.2$

$= -0.5[(x - 4)^2 - 16] + 1.2$

$= -0.5(x - 4)^2 + 8 + 1.2$

$= -0.5(x - 4)^2 + 9.2$

$\therefore$ The greatest height reached by the ball is **9.2 m** and the corresponding horizontal distance travelled is **4 m**.

(i) Substitute $x = 6$ into (1):

$$y = -0.5(6)^2 + 4(6) + 1.2$$
$$= 7.2$$

Since the ball passes through the point (6, 7.2), the ball will hit the balloon.

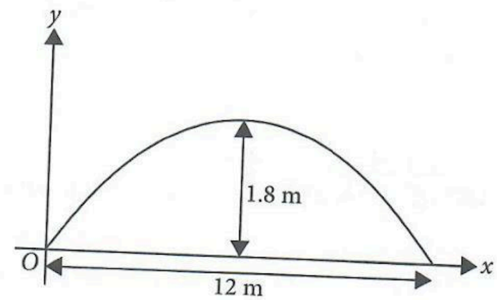
TRY TUTORIAL 1.3: Questions 1, 2

WORKED EXAMPLE 2 **Shape of design structure**

The surface of a play area for children is shaped like a parabola. The play area spans 12 m horizontally and the highest point of the play area is 1.8 m above the ground.

(i) Given that its cross section can be modelled by the equation $y = a(x - p)(x - q)$, where x m is the horizontal distance from a fixed point O and y m is the height of the play area, find the values of the constants a, p and q.

(ii) Two children stand at different points on the play area, each the same distance away from the highest point. Given that one child stands 4.25 m horizontally away from O, find the horizontal distance between the children.



SOLUTION

(i) $y = a(x - p)(x - q)$ — (1)

Substitute $p = 0$, $q = 12$ into (1):

$$y = ax(x - 12) \quad \text{— (2)}$$

Substitute $x = 6$ and $y = 1.8$ into (2):

$$1.8 = a(6)(6 - 12)$$
$$a = -\frac{1}{20}$$

$\therefore$ $\boldsymbol{a = -\frac{1}{20}, p = 0, q = 12}$ or $\boldsymbol{a = -\frac{1}{20}, p = 12, q = 0}$

(ii) $y = -\frac{1}{20}x(x-12)$

By symmetry,

Horizontal distance between the children $= 12 - 2(4.25)$
$= \mathbf{3.5\ m}$

NAME: ______________________ CLASS: ____________ DATE: __________

TUTORIAL **1.3**

1. **Projectile motion.** The height, y metres, of an object released by a machine can be modelled by the equation $y = -0.8x^2 + 8x + 1$, where x is the horizontal distance from the machine in metres.

   (i) Find the greatest height reached by the object and the corresponding horizontal distance travelled.

   (ii) Find the height of the object when it has travelled a horizontal distance of 2 m, stating whether it is moving upward or downward.

2. **Height of a ball.** The height, h metres, of a ball above the ground t seconds after it is kicked from the ground can be modelled by the equation $h = at^2 + bt + c$, where a, b and c are constants.

   (i) State the value of c and give a reason for your answer.

   (ii) Explain why the set of values $a = 2$ and $b = 4$ does not satisfy the equation given in the question.

   (iii) When $a = -0.5$ and $b = 3$, find the time it takes for the ball to return to the ground.

3. **Dimensions of a frame.** The perimeter of a rectangular frame with dimensions x cm by y cm is 1 m.

   (i) Find an expression, in terms of x, for the area, A cm², of the frame.

   (ii) Hence, find the maximum area of the frame, and show that this only happens when the frame is in the shape of a square.

4. **Aiming high and far.** In a game called Aim-High, a player places his token at a fixed point, then chooses one of three given options to release the token. The height, h units, of the token is a function of the horizontal distance travelled, x units. The following shows the options for a particular try:

   If the player chooses option A, his token will travel the highest.

   If the player chooses option B, his token will travel the furthest horizontally.

   If the player chooses option C, his token follows the path $h = -0.25(x - 2)^2 + 4$.

   Suggest an equation for option A and for option B.

IDEAS

NAME: ______________________ CLASS: __________ DATE: __________

Quick Test 1

[Total marks: 15]

1. **(i)** Express $2x^2 - 12x + 3$ in the form $a(x - h)^2 + k$. [1]

(ii) State the coordinates of the turning point of the curve $y = 2x^2 - 12x + 3$. [1]

(iii) The value of y can never be less than p. Write down the value of p. [1]

2. The minimum point of the graph of $y = ax^2 - 6x + b$ lies entirely above the x-axis.

(i) Explain why the set of values $a = 2$ and $b = 11$ satisfies the conditions. [2]

(ii) Show that $a = 11$ and $b = \frac{1}{2}$ does not satisfy the conditions. [3]

3. **Height of toy train track.** The height, y m, of a section of a toy train track above the ground can be modelled by the equation $y = ax^2 + bx + c$, where a, b and c are constants, and x is the horizontal distance of the track from its starting point.

(i) In the case where $a = -0.25$, $b = 0.5$ and $c = 0.5$, explain why the track will not reach a height of 0.8 m. [2]

(ii) Sketch the graph of $y = -0.25x^2 + 0.5x + 0.5$ and write down the coordinates of the turning point. [3]

(iii) Explain the significance of the y-intercept of the graph in part **(ii)**. [1]

(iv) This section of the track spans a width of p m. Given that the height of the track at the starting point is the same as that at the end, state the value of p. [1]