CHAPTER 10 Trigonometric Equations and Identities

Chapter Summary

Trigonometric equations and identities

Principal values:

- $-90° \leqslant \sin^{-1} x \leqslant 90°$, where $-1 \leqslant x \leqslant 1$
- $0° \leqslant \cos^{-1} x \leqslant 180°$, where $-1 \leqslant x \leqslant 1$
- $-90° < \tan^{-1} x < 90°$, where x is any real number

Basic identities:

- $\sin^2 \theta + \cos^2 \theta = 1$
- $1 + \tan^2 \theta = \sec^2 \theta$
- $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$

Addition Formulae:

- $\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\tan (A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Double Angle Formulae:

- $\sin 2A = 2 \sin A \cos A$
- $\cos 2A = \cos^2 A - \sin^2 A$
  $= 2 \cos^2 A - 1$
  $= 1 - 2 \sin^2 A$
- $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

_R_-Formulae

- $a \sin \theta \pm b \cos \theta = R \sin (\theta \pm \alpha)$
- $a \cos \theta \pm b \sin \theta = R \cos (\theta \mp \alpha)$

where $R = \sqrt{a^2 + b^2}$ and $\tan \alpha = \dfrac{b}{a}$

Proving trigonometric identities

use three basic trigonometric identities,
the Addition Formulae and/or
the Double Angle Formulae

Manipulate the more complicated expression on one side of the equation such that it simplifies to the expression on the other side of the equation.

10.1 Trigonometric equations

Objectives Checklist

- State the principal values of $\sin^{-1} x$, $\cos^{-1} x$ or $\tan^{-1} x$
- Solve trigonometric equations

Notes and Worked Examples

Although there are many possible answers to an equation $\sin x = h$ or $\cos x = h$ or $\tan x = k$, for which $-1 \leqslant h \leqslant 1$ and k is any real number, a scientific calculator will only give the principal value.

> Principal values of $\sin^{-1} x$: $-90° \leqslant \sin^{-1} x \leqslant 90°$, where $-1 \leqslant x \leqslant 1$
> Principal values of $\cos^{-1} x$: $0° \leqslant \cos^{-1} x \leqslant 180°$, where $-1 \leqslant x \leqslant 1$
> Principal values of $\tan^{-1} x$: $-90° < \tan^{-1} x < 90°$, where x is any real number

WORKED EXAMPLE 1 Finding the principal values

(a) Find the principal value, in degrees, of $\sin^{-1}\left(-\frac{1}{2}\right)$.

(b) Find the principal value, in radians as a multiple of π, of $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

SOLUTION

(a) $\sin 30° = \frac{1}{2}$

$\sin(-30°) = -\sin 30°$

$= -\frac{1}{2}$

$\therefore$ Principal value of $\sin^{-1}\left(-\frac{1}{2}\right) = \mathbf{-30°}$

(b) $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\cos\left(\pi - \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ ⟵ $\cos(\pi - \theta) = -\cos\theta$

$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

$\therefore$ Principal value of $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \mathbf{\frac{5\pi}{6}}$

TRY TUTORIAL 10.1: Questions 1–5

WORKED EXAMPLE 2 Solving trigonometric equations involving basic angles

Solve each of the following equations.

(a) $\sin x = 0.5$, for $0° \leqslant x \leqslant 360°$

(b) $\cos x = -\tan 30°$, for $-180° < x < 180°$

(c) $\tan 2x = 2$, for $0 \leqslant x \leqslant 2\pi$

(d) $\operatorname{cosec}(x - 10°) = 2.5$, for $0° < x < 270°$

SOLUTION

(a) $\sin x = 0.5$

$\alpha = 30°$

$x = \mathbf{30°, 150°}$

(b) $\cos x = -\tan 30°$

$= -\frac{1}{\sqrt{3}}$

$\alpha = 54.736°$ (to 3 d.p.)

$x = \mathbf{125.3°, -125.3°}$ (to 1 d.p.)

(c) $\tan 2x = 2$

$\alpha = 1.1071$ (to 5 s.f.)

$2x = 1.1071, 4.2487, 7.3903, 10.532$ (to 5 s.f.)

$x = \mathbf{0.554, 2.12, 3.70, 5.27}$ (to 3 s.f.)

(d) $\operatorname{cosec}(x - 10°) = 2.5$

$\sin(x - 10°) = 0.4$

$\alpha = 23.578°$ (to 3 d.p.)

$x - 10° = 23.578°, 156.422°$ (to 3 d.p.)

$x = \mathbf{33.6°, 166.4°}$ (to 1 d.p.)

TRY TUTORIAL 10.1 Questions 6–12

WORKED EXAMPLE 3 Solving trigonometric equations involving squares or factorisation

Solve each of the following equations.

(a) $\sin^2 x = 3\cos^2 x$, for $0° \leqslant x \leqslant 360°$

(b) $2\sin x\cos x = \cos x$, for $0 \leqslant x \leqslant 2\pi$

SOLUTION

(a) $\sin^2 x = 3\cos^2 x$

$\tan^2 x = 3$ ← Since $\frac{\sin x}{\cos x} = \tan x$, then $\frac{\sin^2 x}{\cos^2 x} = \left(\frac{\sin x}{\cos x}\right)^2 = \tan^2 x$.

$\tan x = \pm\sqrt{3}$

$\alpha = 60°$

$x = \mathbf{60°, 120°, 240°, 300°}$

(b) $2\sin x\cos x = \cos x$

$2\sin x\cos x - \cos x = 0$ ← Do not divide both sides by $\cos x$.

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ or $\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\alpha = \frac{\pi}{6}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

TRY TUTORIAL 10.1: Questions 13, 14

NAME: ______________________ CLASS: ____________ DATE: ________

TUTORIAL **10.1**

1. State the values between which each of the following must lie:

(i) the principal value of $\sin^{-1} x$,

(ii) the principal value of $\cos^{-1} x$.

2. Find the principal value, in degrees, of each of the following.

(a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **(b)** $\cos^{-1}\left(-\frac{1}{2}\right)$

(c) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ **(d)** $\tan^{-1}(-\sqrt{3})$

3. Find the principal values, in radians as a multiple of π, of each of the following.

(a) $\sin^{-1}\left(\frac{1}{2}\right)$ **(b)** $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

(c) $\tan^{-1}\sqrt{3}$ **(d)** $\tan^{-1}(-1)$

4. Without using a calculator, find the principal value, in radians as a multiple of π, of each of the following.

(a) $\sin^{-1}\left(\cos\frac{7\pi}{4}\right)$ **(b)** $\cos^{-1}\left[\tan\left(-\frac{5\pi}{4}\right)\right]$

5. It is given that $x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

(i) Express x in terms of π.

(ii) Hence, find the exact value of $\cos 2x + \tan x$.

6. Solve each of the following equations for $0° \leqslant x \leqslant 360°$.

(a) $\sin x = \frac{\sqrt{3}}{2}$ **(b)** $\cos x = 0.5$

(c) $\tan x = 1$ **(d)** $\sin x = -0.4$

(e) $\cos x = -\frac{1}{\sqrt{2}}$ **(f)** $\tan x = -6$

(g) $\sin x = \sin 18°$ **(h)** $\cos x = -\cos 79°$

7. Solve each of the following equations for $0° \leqslant x \leqslant 360°$.

(a) $\sin 2x = -1$

(b) $\cos\frac{1}{2}x = \frac{\sqrt{3}}{2}$

(c) $\tan 3x = 0$

(d) $\sin(x + 40°) = 0.85$

(e) $\cos(x - 16°) = 0.7$

(f) $\tan(2x + 29°) = -1.06$

(g) $\operatorname{cosec}\left(\frac{1}{2}x - 5°\right) = 4$

(h) $\sec(100° - 3x) = 2$

8. Solve each of the following equations.

(a) $6\sin x = 3\cos x$, for $-360° \leqslant x \leqslant 0°$

(b) $\tan(-2x) = 0.9$, for $-270° < x < 0°$

(c) $\sec(1.4x - 20°) = 7$, for $540° \leqslant x \leqslant 720°$

(d) $\cot\left(\frac{2}{3}x + 15°\right) = 0.108$, for $-180° < x < 180°$

9. Solve each of the following equations.

(a) $\cos 2x = -\frac{\sqrt{3}}{2}$, for $0 < x < 2\pi$

(b) $\tan(2x + 1) = 0.5$, for $0 < x < \pi$

(c) $5\sin x + 8\cos x = 0$, for $0 < x < 6$

(d) $\ln\left[\operatorname{cosec}\left(3x - \frac{\pi}{4}\right)\right] = 2$, for $-\frac{\pi}{6} \leqslant x \leqslant \frac{\pi}{6}$

10. (i) On the same axes, sketch the graphs of $y = -3\sin 2x$ and $y = 1$ for $0° \leqslant x \leqslant 180°$.

(ii) Solve the equation $1 + 3\sin 2x = 0$ for $0° \leqslant x \leqslant 180°$, and explain the graphical significance of your answer.

11. The function f is defined, for all values of x, by

$$f(x) = 2\sin\frac{x}{2} + c,$$

where c is a constant. The graph of $y = f(x)$ passes through the point $\left(\frac{5\pi}{3}, -1\right)$.

(i) Find the value of c.

(ii) Solve the equation $f(x) = 0$ for $0 \leqslant x \leqslant 2\pi$ and state the significance of your answer in the context of this question.

12. The equation of a graph is $y = a + b\cos\frac{1}{2}x$, where a and b are positive integers.

(i) State the period of y.

Given that the maximum and minimum values of y are 3 and −5 respectively, find

(ii) the amplitude of y,

(iii) the value of a and of b.

Using the values of a and b found in part **(iii)**,

(iv) find, in degrees, the smallest positive value of x for which $y = 0$,

(v) sketch the graph of y for $-360° \leqslant x \leqslant 180°$.

13. Solve each of the following equations.

(a) $\sin^2 x = \frac{3}{4}$, for $0° \leqslant x \leqslant 360°$

(b) $2\cos^2 x = 1$, for $0° < x < 360°$

(c) $\sec^2 x - 9 = 0$, for $0 \leqslant x \leqslant 2\pi$

(d) $8\cot^2 2x = \tan 2x$, for $0 < x < \pi$

14. Solve each of the following equations.

(a) $2\sin x\cos x = \sqrt{3}\cos x$, for $0° \leqslant x \leqslant 360°$

(b) $\sec^2 x = 5\sec x$, for $180° < x < 540°$

(c) $4(1 + 2\tan^2 x) = 33\tan x$, for $0 \leqslant x \leqslant 2\pi$

(d) $\operatorname{cosec} x - \cot x = 2\cos x\cot x$, for $-\pi < x < \pi$

IDEAS

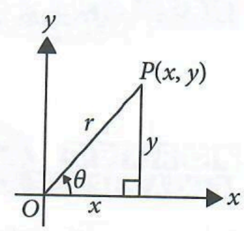
10.2 Trigonometric identities

Objectives Checklist

- State the three basic trigonometric identities
- Apply the trigonometric identities to simplify trigonometric expressions and solve trigonometric equations

Notes and Worked Examples

Consider a right-angled triangle with sides x, y and r as shown.



$$\sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r} \text{ and } \tan\theta = \frac{y}{x}$$

Using Pythagoras' Theorem,

$$x^2 + y^2 = r^2$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad \text{— (1)}$$

Dividing (1) by $\cos^2\theta$,

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta \qquad \text{— (2)}$$

Dividing (1) by $\sin^2\theta$,

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta \qquad \text{— (3)}$$

The three basic trigonometric identities are **equivalent**:

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

WORKED EXAMPLE 1 Simplifying trigonometric expressions using basic identities

Simplify each of the following.

(a) $(\sec\theta + 1)(\sec\theta - 1) + \tan^2\theta$

(b) $\dfrac{\cot^2 x}{1-\sin^2 x} - \operatorname{cosec}^2 x$

SOLUTION

(a) $(\sec\theta + 1)(\sec\theta - 1) + \tan^2\theta$

$= \sec^2\theta - 1 + \tan^2\theta$

$= \tan^2\theta + \tan^2\theta$

$= \mathbf{2\tan^2\theta}$

(b) $\dfrac{\cot^2 x}{1-\sin^2 x} - \operatorname{cosec}^2 x$

$= \dfrac{\frac{\cos^2 x}{\sin^2 x}}{\cos^2 x} - \operatorname{cosec}^2 x$

$= \dfrac{1}{\sin^2 x} - \dfrac{1}{\sin^2 x}$

$= \mathbf{0}$

TRY TUTORIAL 10.2: Questions 1, 2

WORKED EXAMPLE 2 Solving trigonometric equations involving basic identities

Solve each of the following equations.

(a) $2\sin^2\theta = 3\cos\theta$, for $0° \leqslant \theta \leqslant 360°$

(b) $4\sec^2 x = 15\tan x - 5\sin^2 x - 5\cos^2 x$, for $0 < x < \pi$

SOLUTION

(a)

$2\sin^2\theta = 3\cos\theta$

$2(1 - \cos^2\theta) = 3\cos\theta$

$2 - 2\cos^2\theta = 3\cos\theta$

$2\cos^2\theta + 3\cos\theta - 2 = 0$

$(2\cos\theta - 1)(\cos\theta + 2) = 0$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -2$

$\alpha = 60°$ (no solution)

$\theta = 60°, 300°$

$\therefore \mathbf{\theta = 60°, 300°}$

(b)

$4\sec^2 x = 15\tan x - 5\sin^2 x - 5\cos^2 x$

$4\sec^2 x = 15\tan x - 5(\sin^2 x + \cos^2 x)$

$4\sec^2 x = 15\tan x - 5 \leftarrow \sin^2 x + \cos^2 x = 1$

$4(\tan^2 x + 1) = 15\tan x - 5$

$4\tan^2 x + 4 - 15\tan x + 5 = 0$

$4\tan^2 x - 15\tan x + 9 = 0$

$(4\tan x - 3)(\tan x - 3) = 0$

$\tan x = \frac{3}{4}$ or $\tan x = 3$

$\alpha = 0.643\ 50$ (to 5 s.f.) $\alpha = 1.2490$ (to 5 s.f.)

$x = 0.644$ (to 3 s.f.) $x = 1.25$ (to 3 s.f.)

$\therefore \mathbf{x = 0.644, 1.25}$

TRY TUTORIAL 10.2: Questions 3–7

NAME: ______________________ CLASS: ____________ DATE: __________

TUTORIAL 10.2

1. Simplify each of the following.
   (a) $\cot^2\theta - \operatorname{cosec}^2\theta$
   (b) $\dfrac{\sin x - \operatorname{cosec} x}{\cos^2 x}$

2. It is given that $y = 3\sin^2 x + \cos^2 x$. Are the following statements correct? Explain your answer.
   (a) The expression $3\sin^2 x + \cos^2 x$ can be simplified to 3.
   (b) The minimum value of y can never be a negative value.

3. Solve each of the following equations.
   (a) $3\cos^2 x + 17\sin x = 13$, for $0° \leqslant x \leqslant 360°$
   (b) $4\tan^2 x + 2 = 7\sec x$, for $0° < x < 360°$
   (c) $4\cot^2 x + 17\operatorname{cosec} x = 11$, for $0 \leqslant x \leqslant 2\pi$
   (d) $\operatorname{cosec}^2 x = 2\cot x$, for $3\pi < x < 4\pi$

4. Solve each of the following equations.
   (a) $\dfrac{1}{9\sin x + 9\cos x} = \cos x - \sin x$, for $-360° \leqslant x \leqslant 0°$
   (b) $8\sec^2\frac{1}{2}x - 1 = 18\tan\frac{1}{2}x$, for $-2\pi < x < \pi$

5. Find two possible values of x, $7 < x < 12$, that satisfy the equation

$$13\sin^2 x = 12 - \cos x - \cos^2 x.$$

6. (i) Show that $2\sin\theta + 7\cot\theta = 5\operatorname{cosec}\theta$ can be expressed as $2\cos^2\theta - 7\cos\theta + 3 = 0$.
   (ii) Hence, solve the equation $2\sin 2\theta + 7\cot 2\theta = 5\operatorname{cosec} 2\theta$ for $0° < \theta < 360°$.

7. When drawn on the same diagram, the line $y = 2.5$ intersects the curve $y = \operatorname{cosec}^2 x + \cot^2 x$ for values of x between 1 and 6. Find the number of points of intersection.

10.3 Addition Formulae

Objectives Checklist

- State the Addition Formulae
- Apply the Addition Formulae to simplify trigonometric expressions and solve trigonometric equations

Notes and Worked Examples

For two angles A and B, the **Addition Formulae** are:

- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

WORKED EXAMPLE 1 Finding the trigonometric ratios using the Addition Formulae

Without using a calculator, and showing all your working, find the value of sin 75°, giving your answer in the form $\dfrac{\sqrt{a}+\sqrt{b}}{4}$, where a and b are integers.

SOLUTION

$\sin 75° = \sin(45° + 30°)$ ⟵ Write 75° as the sum of two special angles.

$= \sin 45° \cos 30° + \cos 45° \sin 30°$

$= \left(\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{\sqrt{3}}{2}\right) + \left(\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{1}{2}\right)$

$= \dfrac{\sqrt{6}}{4} + \dfrac{\sqrt{2}}{4}$

$= \dfrac{\sqrt{6}+\sqrt{2}}{4}$

TRY TUTORIAL 10.3: Questions 1–5

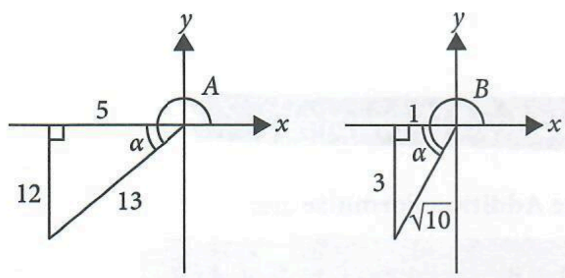
WORKED EXAMPLE 2 **Finding the trigonometric ratios of compound angles**

The angles A and B are such that $\sin A = -\frac{12}{13}$ and $\tan B = 3$, where A and B are in the same quadrant. Find the exact value of each of the following.

(a) $\cos(A - B)$ **(b)** $\tan(A + B)$

SOLUTION

Since $\sin A < 0$ and $\tan B > 0$,
A and B lie in the 3rd quadrant.



(a) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$= \left(-\frac{5}{13}\right)\left(-\frac{1}{\sqrt{10}}\right) + \left(-\frac{12}{13}\right)\left(-\frac{3}{\sqrt{10}}\right)$$

$$= \frac{5+36}{13\sqrt{10}}$$

$$= \frac{41}{13\sqrt{10}}$$

(b) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$$= \frac{\frac{12}{5}+3}{1-\left(\frac{12}{5}\right)(3)}$$

$$= -\frac{27}{31}$$

TRY TUTORIAL 10.3: Questions 6–11

WORKED EXAMPLE 3 **Solving trigonometric equations using the Addition Formulae**

Find all the angles between 0° and 360° inclusive which satisfy the equation $\cos(x + 30°) = 2 \sin x$.

SOLUTION

$$\cos(x + 30°) = 2 \sin x$$

$$\cos x \cos 30° - \sin x \sin 30° = 2 \sin x \longleftarrow \cos(x + 30°) \neq \cos x + \cos 30°$$

$$\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = 2 \sin x$$

$$\sqrt{3}\cos x - \sin x = 4 \sin x$$

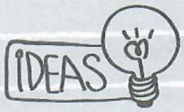
$$5 \sin x = \sqrt{3}\cos x$$

$$\tan x = \frac{\sqrt{3}}{5}$$

$$\alpha = 19.107° \text{ (to 3 d.p.)}$$

$$x = \mathbf{19.1°, 199.1°} \text{ (to 1 d.p.)}$$

TRY TUTORIAL 10.3: Questions 12–16


IDEAS

NAME: ______________________ **CLASS:** __________ **DATE:** __________

TUTORIAL 10.3

1. Without using a calculator, find the exact value of each of the following.

(a) $\sin 59° \cos 14° - \cos 59° \sin 14°$ **(b)** $\cos 220° \cos 70° + \sin 220° \sin 70°$

(c) $\dfrac{\tan 68° - \tan 23°}{1 + \tan 68° \tan 23°}$ **(d)** $\dfrac{1 - \tan 76° \tan 44°}{\tan 76° + \tan 44°}$

2. Without using a calculator, show that

(i) $\cos 75° = \dfrac{\sqrt{6}-\sqrt{2}}{4}$, **(ii)** $\sin^2 75° = \dfrac{2+\sqrt{3}}{4}$.

3. Without using a calculator, show that

(i) $\tan 15° = 2-\sqrt{3}$, **(ii)** $\sec^2 15° = 4 \tan 15°$.

4. Show that $\sin 165° - \cos 165°$ can be written in the form $\dfrac{\sqrt{k}}{2}$, where k is an integer.

5. A, B and C are the angles of a triangle.

(i) Show that $\cos (A + B) = -\cos C$.

Angle $A = \dfrac{\pi}{3}$ radians and angle $B = \dfrac{\pi}{4}$ radians.

(ii) Without using a calculator, show that $\cos C$ can be expressed in the form $\dfrac{\sqrt{a}-\sqrt{b}}{4}$, where a and b are integers.

6. The angles A and B are such that $\cos A = -\dfrac{24}{25}$ and $\tan B = -2\sqrt{2}$, where A and B are in the same quadrant. Find the exact value of each of the following.

(a) $\tan A \sec B$ **(b)** $\sin (A + B)$

(c) $\cos (A - B)$ **(d)** $\tan (A - B)$

7. A and B are acute angles such that $\cos A = \dfrac{2}{\sqrt{29}}$ and $\sin B = \dfrac{3}{5}$. Show that

(i) $\tan (A - B) = \dfrac{14}{23}$,

(ii) $\cos (A + B)$ can be expressed in the form $\dfrac{k\sqrt{29}}{145}$, and state the value of k.

8. It is given that $\cos A = -\dfrac{12}{13}$ and $\tan B = \dfrac{1}{4}$, where A and B are positive angles in different quadrants and $0 < A + B < \pi$.

(i) Determine the quadrants in which angles A and B lie and explain your answer.

(ii) Hence, find the exact value of $\sin (B - A)$.

9. A and B are acute angles such that $\sin(A - B) = \frac{1}{2}$ and $\sin A \cos B = \frac{7}{10}$. Without using a calculator, find the exact value of

(i) $\cos A \sin B$,

(ii) $\sin(A + B)$,

(iii) $\frac{\tan A}{\tan B}$.

10. Angles A and B are such that $\sin(A - B) = 4\cos(A + B)$.

(i) Show that $\tan B = \frac{4 - \tan A}{4\tan A - 1}$.

(ii) Given that $A = \frac{\pi}{6}$, find, without using a calculator, $\tan B$ in the form $\frac{h\sqrt{3} + k}{13}$, where h and k are integers.

11. It is given that $\sin x \sin y = \frac{1}{12}$ and $\cos x \cos y = \frac{3}{4}$.

(i) Find the value of $\cos(x + y)$ and of $\cos(x - y)$.

(ii) Hence, find the acute angles x and y.

12. Solve each of the following equations.

(a) $4\cos(x - 30°) = 3\sin x$, for $0° \leqslant x \leqslant 360°$

(b) $5\sin\left(2x + \frac{2\pi}{3}\right) = \cos\left(2x - \frac{\pi}{6}\right)$, for $-\pi < x < \pi$

13. Given that $\tan(x - 60°) - \tan(x + 60°) = 3$, where x is obtuse, find the value of x.

14. (i) Henry says that the expressions $\sin\left(x + \frac{\pi}{6}\right)$ and $\cos\left(\frac{\pi}{3} - x\right)$ are equivalent. Do you agree? Explain your answer.

(ii) Solve the equation $\sin\left(x + \frac{\pi}{6}\right) + 4\cos\left(\frac{\pi}{3} - x\right) - 1 = 0$ for $-3 < x < 3$.

15. (i) Using $\cos 3x = \cos(2x + x)$, $\sin 2x = 2\sin x \cos x$ and $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$, show that $\cos 3x$ may be expressed as $4\cos^3 x - 3\cos x$.

(ii) Find all the values of x between 0° and 360° inclusive for which $3\cos 3x = 23\cos^2 x$.

16. The equations of two straight lines l_1 and l_2 are $y = ax + 3$ and $y = bx - 2$ respectively. In the case where $a = 2$ and $b = -\frac{1}{2}$, use the Addition Formula to show that the lines meet at right angles.

10.4 Double Angle Formulae

Objectives Checklist

- State the Double Angle Formulae
- Apply the Double Angle Formulae to simplify trigonometric expressions and solve trigonometric equations

Notes and Worked Examples

The **Double Angle Formulae** are:

- $\sin 2A = 2 \sin A \cos A$
- $\cos 2A = \cos^2 A - \sin^2 A$

  $= 2 \cos^2 A - 1$

  $= 1 - 2 \sin^2 A$
- $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

WORKED EXAMPLE 1 **Finding the trigonometric ratios using the Double Angle Formulae**

It is given that $\sin A = \frac{3}{5}$ and $\tan A < 0$. Without using a calculator, find the value of each of the following.

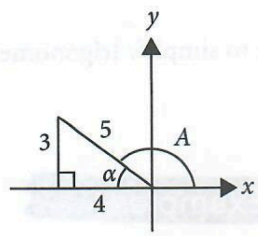
(a) $\cos 2A$

(b) $\cos \frac{A}{2}$

SOLUTION

Since $\sin A > 0$ and $\tan A < 0$,

A lies in the 2nd quadrant.



(a) $\cos 2A = 1 - 2\sin^2 A$

$$= 1-2\left(\frac{3}{5}\right)^2$$

$$= \mathbf{\frac{7}{25}}$$

(b)

$$\cos A = 2\cos^2\frac{A}{2}-1$$

$$-\frac{4}{5} = 2\cos^2\frac{A}{2}-1$$

$$2\cos^2\frac{A}{2} = \frac{1}{5}$$

$$\cos^2\frac{A}{2} = \frac{1}{10}$$

$$\cos\frac{A}{2} = \pm\frac{1}{\sqrt{10}}$$

$$= \pm\frac{\sqrt{10}}{10}$$

Since $90° < A < 180°$, then $45° < \frac{A}{2} < 90°$.

$$\therefore \cos\frac{A}{2} = \mathbf{\frac{\sqrt{10}}{10}}$$

TRY TUTORIAL 10.4: Questions 1–6

WORKED EXAMPLE 2 **Solving trigonometric equations using the Double Angle Formulae**

Solve each of the following equations.

(a) $\sin 2x - \sin x = 0$, for $0° \leqslant x \leqslant 360°$

(b) $8 \sin x \cos x = -3$, for $0 \leqslant x \leqslant 2\pi$

(c) $2 \cos 2x - 3 \cos x + 1 = 0$, for $0° < x < 270°$

(d) $\tan 2x - 5 \tan x = 0$, for $0 < x < \frac{3\pi}{2}$

SOLUTION

(a)

$$\sin 2x - \sin x = 0$$

$$2 \sin x \cos x - \sin x = 0$$ ← Do not divide both sides by $\sin x$.

$$\sin x (2 \cos x - 1) = 0$$

$\sin x = 0$ or $\cos x = \frac{1}{2}$

$x = 0°, 180°, 360°$ $\alpha = 60°$

$x = 60°, 300°$

$\therefore$ $x =$ **0°, 60°, 180°, 300°, 360°**

(b)

$$8 \sin x \cos x = -3$$

$$4(2 \sin x \cos x) = -3$$

$$\sin 2x = -\frac{3}{4}$$

$$\alpha = 0.848\ 06 \text{ (to 5 s.f.)}$$

$$2x = 3.9897, 5.4351, 10.273, 11.718 \text{ (to 5 s.f.)}$$ ← Since $0 \leqslant x \leqslant 2\pi$, then $0 \leqslant 2x \leqslant 4\pi$.

$x =$ **1.99, 2.72, 5.14, 5.86** (to 3 s.f.)

(c)

$$2 \cos 2x - 3 \cos x + 1 = 0$$

$$2(2 \cos^2 x - 1) - 3 \cos x + 1 = 0$$

$$4 \cos^2 x - 2 - 3 \cos x + 1 = 0$$

$$4 \cos^2 x - 3 \cos x - 1 = 0$$

$$(4 \cos x + 1)(\cos x - 1) = 0$$

$\cos x = -\frac{1}{4}$ or $\cos x = 1$ (rejected)

$\alpha = 75.522°$ (to 3 d.p.)

$x =$ **104.5°, 255.5°** (to 1 d.p.)

(d)

$$\tan 2x - 5\tan x = 0$$

$$\frac{2\tan x}{1-\tan^2 x} - 5\tan x = 0$$

$$2\tan x - 5\tan x + 5\tan^3 x = 0$$

$$5\tan^3 x - 3\tan x = 0$$

$$\tan x\,(5\tan^2 x - 3) = 0$$

$\tan x = 0$ or $\tan x = \pm\sqrt{\frac{3}{5}}$

$x = \pi$ $\qquad$ $\alpha = 0.659\ 06$ (to 5 s.f.)

$x = 0.659, 2.48, 3.80$ (to 3 s.f.)

$\therefore$ **$x = 0.659, 2.48, \pi, 3.80$**

TRY TUTORIAL 10.4: Questions 7–10

NAME: ______________________ CLASS: __________ DATE: __________

TUTORIAL 10.4

1. Without using a calculator, find the exact value of each of the following.

(a) $2 \sin 22.5° \cos 22.5°$

(b) $3 - 6\sin^2 \frac{\pi}{12}$

(c) $\frac{4 \tan 75°}{1 - \tan^2 75°}$

(d) $\left(\cos\frac{\pi}{8} + \sin\frac{\pi}{8}\right)^2$

2. θ is an acute angle such that $\cos \theta = \frac{4}{5}$. Find the value of each of the following.

(a) $\sin 2\theta$

(b) $\cos 2\theta$

(c) $\tan 2\theta$

(d) $\cos 4\theta$

3. Given that $\operatorname{cosec} A = \frac{k}{3}$, where A is acute, express each of the following in terms of k.

(a) $\cot A$

(b) $\sec 2A$

4. The acute angles A and B are such that $\tan (A - B) = \frac{1}{2}$ and $\sin A = \frac{3}{5}$. Without using a calculator, find the exact value of

(a) $\cos 2A$,

(b) $\tan 2B$,

(c) $\cos 4A$,

(d) $\sin\frac{1}{2}A$.

5. Given that $p = \sin x + \cos x$ and $q = \sin x - \cos x$, show that $p^2 + q^2$ is independent of x.

6. **(i)** Show that $6 \sin^2 x + 4 \cos^2 x$ can be written as $a + b \cos 2x$, where a and b are integers.

Hence

(ii) state the period and amplitude of $6 \sin^2 x + 4 \cos^2 x$,

(iii) sketch the graph of $y = 6 \sin^2 x + 4 \cos^2 x$ for $0 \leqslant x \leqslant 2\pi$ radians.

7. Solve each of the following equations.

(a) $\cos x \sin x = 0.5$, for $0° < x < 360°$

(b) $\sin^2 x - \cos^2 x = 0.234$, for $0 < x < \pi$

(c) $5 \cos 2x - \sin x = 3$, for $-180° \leqslant x \leqslant 270°$

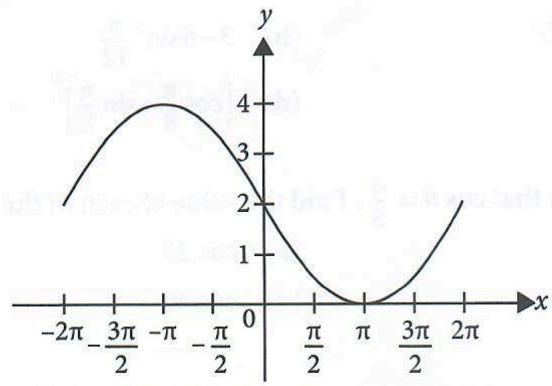
(d) $\tan 4x + \tan 2x = 0$, for $-\pi \leqslant x \leqslant 0$

8. Show that the equation $3 \cos 2x = 19 \cos x - 18$ has no solutions.

9. **(i)** Express $\frac{1}{2}\sin 2\theta(\text{cosec }\theta - \cot\theta)$ as a quadratic expression in $\cos\theta$.

(ii) Use your answer in part **(i)** to find, for $0 \leqslant \theta \leqslant 2\pi$, the exact solutions of the equation $\frac{1}{2}\sin 2\theta(\text{cosec }\theta - \cot\theta) = 2 + 4\cos\theta - 3\cos^2\theta$.

10.



The diagram shows the trigonometric graph $y = a\sin bx + c$, where a, b and c are constants.

(i) Find the values of a, b and c.

(ii) On the same axes, insert the graph of $y = 2\cos x + 2$ for $-2\pi \leqslant x \leqslant 2\pi$.

(iii) Show that the x-coordinates of the points of intersection of the two graphs are satisfied by the equation $2\sin^2\frac{1}{2}x - \sin\frac{1}{2}x - 1 = 0$.

(iv) Hence, find the x-coordinates of the points of intersection of the two graphs.

10.5 Proving of identities

Objectives Checklist

- Prove other trigonometric identities

Notes and Worked Examples

An identity is an equation that is true for all values of the variable that is substituted into both sides of the equation.

The three basic trigonometric identities, the Addition Formulae and the Double Angle Formulae may be used to manipulate the more complicated expression on one side of the equation such that it simplifies to the expression on the other side of the equation.

WORKED EXAMPLE 1 Proving trigonometric identities using the three basic identities

Prove each of the following identities.

(a) $\sec x - \cos x = \sin x \tan x$

(b) $\dfrac{\cos x}{1-\cos x} = \dfrac{\sec x+1}{\tan^2 x}$

SOLUTION

(a) LHS $= \sec x - \cos x$

$= \dfrac{1}{\cos x} - \cos x$

$= \dfrac{1-\cos^2 x}{\cos x}$

$= \dfrac{\sin^2 x}{\cos x}$

$= \sin x\left(\dfrac{\sin x}{\cos x}\right)$

$= \sin x \tan x$

= RHS (proven)

(b) RHS $= \dfrac{\sec x+1}{\tan^2 x}$

$= \dfrac{\sec x+1}{\sec^2 x-1}$

$= \dfrac{\sec x+1}{(\sec x+1)(\sec x-1)}$

$= \dfrac{1}{\sec x-1}$

$= \dfrac{1}{\dfrac{1}{\cos x}-1}$

$= \dfrac{\cos x}{1-\cos x}$

= LHS (proven)

TRY TUTORIAL 10.5: Questions 1, 5–7

WORKED EXAMPLE 2 **Proving trigonometric identities using the Addition Formulae and the Double Angle Formulae**

Prove each of the following identities.

(a) $\dfrac{\sin(A-B)}{\cos A\cos B} = \tan A - \tan B$ **(b)** $\operatorname{cosec} x - \sec x = \dfrac{2\sin x}{1-\cos 2x} - \dfrac{2\cos x}{1+\cos 2x}$

SOLUTION

(a) $\text{LHS} = \dfrac{\sin(A-B)}{\cos A\cos B}$ ⟵ The LHS contains the more complicated expression.

$= \dfrac{\sin A\cos B - \cos A\sin B}{\cos A\cos B}$

$= \dfrac{\sin A\cos B}{\cos A\cos B} - \dfrac{\cos A\sin B}{\cos A\cos B}$

$= \dfrac{\sin A}{\cos A} - \dfrac{\sin B}{\cos B}$

$= \tan A - \tan B$

= RHS (proven)

(b) $\text{RHS} = \dfrac{2\sin x}{1-\cos 2x} - \dfrac{2\cos x}{1+\cos 2x}$ ⟵ The RHS contains the more complicated expression.

$= \dfrac{2\sin x}{1-(1-2\sin^2 x)} - \dfrac{2\cos x}{1+(2\cos^2 x-1)}$

$= \dfrac{2\sin x}{2\sin^2 x} - \dfrac{2\cos x}{2\cos^2 x}$

$= \dfrac{1}{\sin x} - \dfrac{1}{\cos x}$

$= \operatorname{cosec} x - \sec x$

= LHS (proven)

TRY TUTORIAL 10.5: Questions 2–4, 8

NAME: ______________________ **CLASS:** ____________ **DATE:** __________

TUTORIAL 10.5

1. Prove each of the following.

(a) $\cot\theta + \tan\theta = \sec\theta \operatorname{cosec}\theta$

(b) $1 + \sec\theta = \dfrac{\sin\theta\tan\theta}{1 - \cos\theta}$

(c) $\dfrac{\sin\theta}{1 + \cos\theta} = \operatorname{cosec}\theta - \cot\theta$

(d) $\operatorname{cosec}^2\theta = \dfrac{1}{2(1 + \cos\theta)} + \dfrac{1}{2(1 - \cos\theta)}$

(e) $(\operatorname{cosec}\theta - \cot\theta)^2 = \dfrac{1 - \cos\theta}{1 + \cos\theta}$

(f) $\dfrac{\cos\theta}{1 - \sec\theta} + \dfrac{1}{1 - \cos\theta} = 1 + \cos\theta$

2. Using the Addition Formulae, prove each of the following.

(a) $\sin\left(\dfrac{\pi}{2} - x\right) = \cos x$

(b) $\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$

(c) $\sin\left(x + \dfrac{\pi}{2}\right) = \cos x$

(d) $\cos\left(x + \dfrac{\pi}{2}\right) = -\sin x$

(e) $\sin(x + \pi) = -\sin x$

(f) $\cos(x - 2\pi) = \cos x$

(g) $\tan(x + \pi) = \tan x$

(h) $\tan(2\pi - x) = -\tan x$

3. Prove each of the following.

(a) $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

(b) $\dfrac{\cos 2x + 1}{2\sin x\cos x} = \cot x$

(c) $\dfrac{1 + 2\sin x + 2\sin^2 x + \cos 2x}{2\cos x(1 + \sin x)} = \sec x$

(d) $\tan x(\cot 2x + \operatorname{cosec} 2x) = 1$

4. Prove each of the following.

(a) $\cot(x - y) = \dfrac{\cot x\cot y + 1}{\cot y - \cot x}$

(b) $1 - \tan^2 x\tan^2 y = \dfrac{\cos(x + y)\cos(x - y)}{(1 - \sin^2 x)(1 - \sin^2 y)}$

(c) $\dfrac{\sin(x + y) - \sin(x - y)}{\cos(x - y)} = \dfrac{2\tan y}{1 + \tan x\tan y}$

(d) $\cot^2\left(\dfrac{3\pi}{4} - x\right) = \dfrac{\sec^2 x - 2\tan x}{\sec^2 x + 2\tan x}$

5. (i) Show that $\dfrac{\operatorname{cosec} x - \sec x}{\operatorname{cosec} x + \sec x} = \dfrac{1 - \tan x}{1 + \tan x}$.

(ii) Hence find, for $0 \leqslant x \leqslant 2\pi$, the values of x in radians for which $\dfrac{\operatorname{cosec} x - \sec x}{\operatorname{cosec} x + \sec x} = \dfrac{3}{5}$.

6. (i) Show that $\dfrac{\sec^2 \theta + 2 \tan \theta}{1 - \tan^2 \theta} = \dfrac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta}$.

(ii) Give an example of the value of θ in the range $6 < \theta < 10$ for which $\dfrac{\sec^2 \theta + 2 \tan \theta}{\tan^2 \theta - 1} = 3$.

7. (i) Prove that $(\sin x - \operatorname{cosec} x) \times (\cos x - \sec x) \times (\tan x + \cot x) = \cot x$.

(ii) Hence, state the number of solutions of the equation $(\sin x - \operatorname{cosec} x) \times (\cos x - \sec x) \times (\tan x + \cot x) = 3$ in the interval $0 < x < \pi$.

8. (i) Show that $\sin 2x = \dfrac{2}{\cot x + \tan x}$.

(ii) Hence, solve the equation $\dfrac{4}{\cot^2 x + \tan^2 x + 2} = \dfrac{5}{\cot x + \tan x} - 1$ for $0° \leqslant x \leqslant 360°$.

10.6 R-Formulae

Objectives Checklist

- State the R-Formulae
- Apply the R-Formulae to simplify trigonometric expressions and solve trigonometric equations of the type $a \sin \theta \pm b \cos \theta = c$ and $a \cos \theta \pm b \sin \theta = c$

Notes and Worked Examples

The Addition Formula for sin $(A + B)$ is:

$$\sin (A + B) = \sin A \cos B + \cos A \sin B \quad \text{— (1)}$$

Substitute $A = \theta$ and $B = \alpha$ into (1):

$$\sin (\theta + \alpha) = \sin \theta \cos \alpha + \cos \theta \sin \alpha$$

Let $a \sin \theta + b \cos \theta = R \sin (\theta + \alpha)$.

$$a \sin \theta + b \cos \theta = R \sin \theta \cos \alpha + R \cos \theta \sin \alpha$$

$$= (R \cos \alpha) \sin \theta + (R \sin \alpha) \cos \theta \quad \text{— (2)}$$

Comparing coefficients of $\sin \theta$,

$$a = R \cos \alpha \quad \text{— (3)}$$

Comparing coefficients of $\cos \theta$,

$$b = R \sin \alpha \quad \text{— (4)}$$

$(3)^2 + (4)^2$: $a^2 + b^2 = R^2 \cos^2 \alpha + R^2 \sin^2 \alpha$

$$= R^2(\cos^2 \alpha + \sin^2 \alpha)$$

$$R^2 = a^2 + b^2$$

$$R = \sqrt{a^2 + b^2}$$

$(4) \div (3)$: $\dfrac{b}{a} = \dfrac{\sin \alpha}{\cos \alpha}$

$$\tan \alpha = \frac{b}{a}$$

To solve trigonometric equations of the type $a \sin \theta \pm b \cos \theta = c$ and $a \cos \theta \pm b \sin \theta = c$, apply the **$R$-Formulae:**

- $a \sin \theta \pm b \cos \theta = R \sin (\theta \pm \alpha)$
- $a \cos \theta \pm b \sin \theta = R \cos (\theta \mp \alpha)$,

where $R = \sqrt{a^2 + b^2}$ and $\tan \alpha = \dfrac{b}{a}$

The R-Formulae can be used to determine the maximum and minimum values of a trigonometric expression in the form $a \sin \theta \pm b \cos \theta$ and $a \cos \theta \pm b \sin \theta$:

- For $a \sin \theta \pm b \cos \theta$,

  Maximum value = $\sqrt{a^2+b^2}$ when $\sin (\theta \pm \alpha) = 1$

  Minimum value = $-\sqrt{a^2+b^2}$ when $\sin (\theta \pm \alpha) = -1$

- For $a \cos \theta \pm b \sin \theta$,

  Maximum value = $\sqrt{a^2+b^2}$ when $\cos (\theta \mp \alpha) = 1$

  Minimum value = $-\sqrt{a^2+b^2}$ when $\cos (\theta \mp \alpha) = -1$

WORKED EXAMPLE 1

Expressing a trigonometric expression using the R-Formula

(i) Express $3 \sin \theta + 4 \cos \theta$ in the form $R \sin (\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$.

(ii) Hence, solve the equation $3 \sin \theta + 4 \cos \theta = 2$ for $0° \leqslant \theta \leqslant 360°$.

SOLUTION

(i) Let $3 \sin \theta + 4 \cos \theta = R \sin (\theta + \alpha)$.

$R = \sqrt{3^2+4^2}$ ⟵ $a = 3, b = 4$

$= 5$

$\tan \alpha = \frac{4}{3}$

$\alpha = 53.130°$ (to 3 d.p.)

$\therefore 3 \sin \theta + 4 \cos \theta = \mathbf{5 \sin (\theta + 53.130°)}$

(ii) $3 \sin \theta + 4 \cos \theta = 2$

$5 \sin (\theta + 53.130°) = 2$

$\sin (\theta + 53.130°) = \frac{2}{5}$

Basic angle = 23.578° (to 3 d.p.)

$\theta + 53.130° = 156.422°, 383.578°$ (to 3 d.p.)

$\theta = \mathbf{103.3°, 330.4°}$ (to 1 d.p.)

TRY TUTORIAL 10.6: Questions 1, 3–6

WORKED EXAMPLE 2 **Finding the maximum and minimum values using the *R*-Formula**

(i) Express $P = 2.5 \cos \theta + 6 \sin \theta$ in the form $R \cos (\theta - \alpha)$, where R is a positive constant and $0 < \alpha < \frac{\pi}{2}$ radians.

(ii) Deduce the maximum and minimum values of P, and the corresponding values of θ.

SOLUTION

(i) Let $2.5 \cos \theta + 6 \sin \theta = R \cos (\theta - \alpha)$.

$R = \sqrt{2.5^2 + 6^2}$ ⟵ $a = 2.5, b = 6$

$= 6.5$

$\tan \alpha = \frac{6}{2.5}$

$\alpha = 1.1760$ (to 5 s.f.)

$\therefore$ $\mathbf{P = 6.5 \cos (\theta - 1.1760)}$

(ii) Maximum value of $P = \mathbf{6.5}$, when $\cos (\theta - 1.1760) = 1$

$\theta - 1.1760 = 0$

$\theta = \mathbf{1.18}$ (to 3 s.f.)

Minimum value of $P = \mathbf{-6.5}$, when $\cos (\theta - 1.1760) = -1$

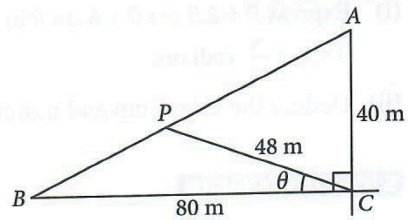
$\theta - 1.1760 = \pi$

$\theta = \mathbf{4.32}$ (to 3 s.f.)

TRY TUTORIAL 10.6: Question 2

WORKED EXAMPLE 3 Applying the *R*-Formula in real-world contexts

Three children, standing at A, B and C respectively, are playing in an open space such that $AC = 40$ m, $BC = 80$ m and angle $ACB = 90°$. When a whistle is blown, they run towards P, a point along AB. The fastest child runs 48 m in a straight line from C to P and angle $PCB = \theta°$.



(i) Express the shortest distance of P from AC and from BC in terms of θ.

(ii) Show that $3\cos\theta + 6\sin\theta = 5$.

(iii) Express $3\cos\theta + 6\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and α is acute.

(iv) Find the value of θ.

SOLUTION

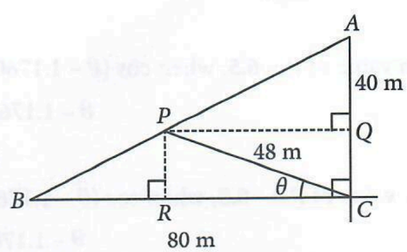
(i) $\cos\theta = \frac{PQ}{48}$

$PQ = 48\cos\theta$

$\sin\theta = \frac{PR}{48}$

$PR = 48\sin\theta$

$\therefore$ Shortest distance of P from AC is **$48\cos\theta$ m**, shortest distance of P from BC is **$48\sin\theta$ m**



(ii) Area of $\triangle ABC$ = Area of $\triangle ACP$ + area of $\triangle BCP$

$$\frac{1}{2}(80)(40) = \frac{1}{2}(40)(48\cos\theta) + \frac{1}{2}(80)(48\sin\theta)$$

$$3200 = 1920\cos\theta + 3840\sin\theta$$

$3\cos\theta + 6\sin\theta = 5$ (shown)

(iii) Let $3\cos\theta + 6\sin\theta = R\cos(\theta - \alpha)$.

$R = \sqrt{3^2 + 6^2}$ $\quad a = 3, b = 6$

$= \sqrt{45}$

$= 3\sqrt{5}$

$\tan\alpha = \frac{6}{3}$

$\alpha = 63.435°$ (to 3 d.p.)

$\therefore 3\cos\theta + 6\sin\theta = \mathbf{3\sqrt{5}\cos(\theta - 63.435°)}$

(iv) $3\cos\theta + 6\sin\theta = 5$

$3\sqrt{5}\cos(\theta - 63.435°) = 5$

$\cos(\theta - 63.435°) = \frac{5}{3\sqrt{5}}$

Basic angle $= 41.810°$ (to 3 d.p.)

$\theta - 63.435° = -41.810°$

$\theta = \mathbf{21.6°}$ (to 1 d.p.)

TRY TUTORIAL 10.6: Questions 7–9

NAME: ______________________ CLASS: __________ DATE: __________

TUTORIAL 10.6

1. Express each of the following as a single trigonometric function.

(a) $4\sin\theta + 3\cos\theta$

(b) $\sin\theta - 5\cos\theta$

(c) $\frac{1}{2}\cos\theta + \frac{1}{2}\sin\theta$

(d) $2\sqrt{2}\cos\theta - \sin\theta$

2. State the maximum and minimum values of each function, and the corresponding values of θ, where $0° < \theta < 360°$.

(a) $5\sin\theta + 12\cos\theta$

(b) $24\cos\theta - 7\sin\theta$

(c) $(\sqrt{3}\sin\theta - \cos\theta)^2$

(d) $\frac{1}{6\cos\theta + 8\sin\theta + 11}$

3. Solve each of the following equations.

(a) $\sin\theta - \cos\theta = 1$, for $0° \leqslant \theta \leqslant 360°$

(b) $4\cos\theta - 9\sin\theta = 3$, for $0 \leqslant \theta \leqslant 2\pi$

4. Aaron and Brian solved the equation $8\cos x + 7\sin x = 6$, for $0 < x < 6$.

(a) Aaron expressed $8\cos x + 7\sin x$ in the form $R\cos(x - \alpha)$, before solving the equation. Show Aaron's working.

(b) Brian expressed $8\cos x + 7\sin x$ in the form $R\sin(x + \alpha)$, before solving the equation. Show Brian's working.

5. Solve the equation $3\tan x + 4 = 2\sec x$ for $0° \leqslant x \leqslant 270°$.

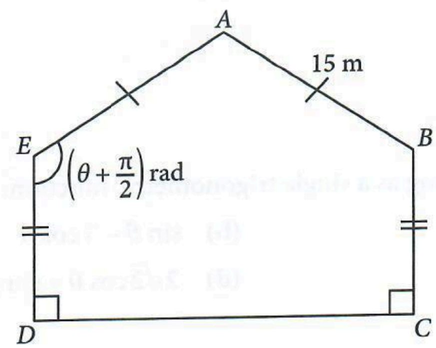
6. **(i)** Write $\cos 2x + \sin 2x$ in the form $R\cos(2x - \alpha)$, where α is acute and $R > 0$.

(ii) Using appropriate identities and the result from part **(i)**, show that $y = 4\sin x\cos x + 4\cos^2 x$ may be written in the form

$$y = 2 + 2\sqrt{2}\cos(2x - 45°).$$

(iii) Hence, find the smallest positive value of x for which $y = 0$.

7.



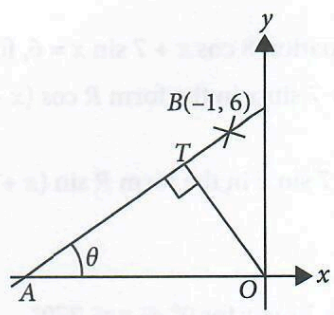
Dimensions of a flower garden. The diagram shows the plan view of a flower garden, $ABCDE$. A fence is constructed around the perimeter of the garden. $AB = 15$ m and angle $AED = \left(\theta + \frac{\pi}{2}\right)$ radians. The perpendicular distance from A to DC is twice of BC.

(i) Show that L m, the length of fencing needed, can be expressed in the form $p + q \sin \theta + r \cos \theta$, where p, q and r are constants.

(ii) Express L in the form $h + k \cos (\theta - \alpha)$, where $k > 0$ and α is an acute angle.

(iii) A contractor estimates that 70 m of fencing is required. Find the possible values of θ.

8.



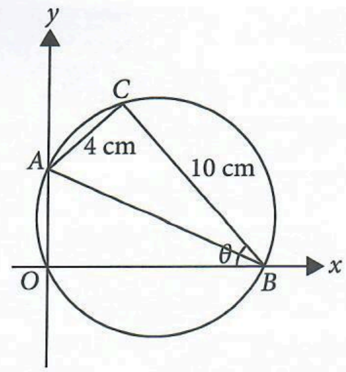
The diagram shows a line passing through the points A and $B(-1, 6)$. The point T lies on AB such that OT is perpendicular to AB.

(i) Given that angle $BAO = \theta$ radians, show that $OT = \sin \theta + 6 \cos \theta$.

(ii) Hence, express OT in the form $R \cos (\theta - \alpha)$, where $R > 0$ and α is an acute angle.

(iii) Find the greatest possible length of OT and the corresponding value of θ.

9.



The diagram shows a circle with diameter AB. The point C lies on the circumference of the circle such that angle $CBO = \theta$ radians. $AC = 4$ cm and $BC = 10$ cm.

(i) Show that $OB = 4 \sin \theta + 10 \cos \theta$.

(ii) Hence, find the greatest possible length of OB and the corresponding value of θ.

IDEAS

NAME: ______________________ CLASS: ____________ DATE: ________

Quick Test 10

[Total marks: 35]

1. (i) Find, in radians, the two principal values of x for which $5\tan^2 x + 24\tan x - 5 = 0$. [4]

(ii) Using a suitable identity, explain why the values of x in part (i) satisfy the equation $\tan 2x = \frac{5}{12}$. [2]

2. (i) Without using a calculator, write down and simplify the expansion of $\cos\left(\theta - \frac{\pi}{2}\right)$ and of $\sin\left(\theta + \frac{\pi}{6}\right)$, leaving each expression in exact form. [3]

(ii) Given that $2\cos\left(\theta - \frac{\pi}{2}\right) = \sin\left(\theta + \frac{\pi}{6}\right)$, show that $\tan\theta = \frac{4+\sqrt{3}}{13}$. [3]

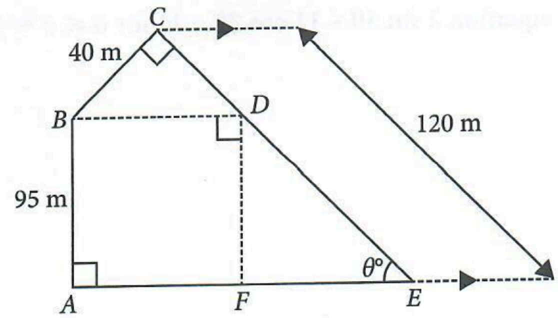
3. (i) Prove that $\dfrac{1}{(1+\sec\theta)(\operatorname{cosec}\theta-\cot\theta)}=\cot\theta$. [4]

(ii) Find, in radians, the acute angle for which $\dfrac{1}{3(1+\sec\theta)(\operatorname{cosec}\theta-\cot\theta)}=\tan\theta$. [2]

4. **(i)** By writing $\sin 3\theta$ as $\sin (2\theta + \theta)$, show that $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$. [3]

(ii) Hence, solve the equation $2 \sin 3\theta + 11 \cos 2\theta = 11$ for $0 \leqslant \theta \leqslant 2\pi$. [4]

5.



Slide at a theme park. The diagram represents the path of a child at a theme park. From A, she takes a vertical lift to B, then climbs 40 m along a ropeway to reach C. From C, she slides 120 m down a slope inclined at $\theta°$ to the horizontal to finally reach E after passing through the point D.

(i) Given that angle $BCD = 90°$, express CD in terms of $\tan\theta$ and hence obtain an expression for DE. [2]

(ii) Hence, show that DF can be expressed in the form $a \sin\theta - b\cos\theta$, where a and b are constants. [3]

(iii) By expressing DF in the form $R\sin(\theta - \alpha)$, find the value of θ. [5]