CHAPTER 11 Gradients, Derivatives and Differentiation Techniques

Chapter Summary

Derivative of y

Gradient of curve at $x = x_0$ is the value of $\frac{dy}{dx}$ at $x = x_0$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(k) = 0$

Constant Multiple Rule: $\frac{d}{dx}[kf(x)] = k\frac{d}{dx}[f(x)]$

Sum/Difference Rule: $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$

Chain Rule: $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}\frac{d}{dx}[f(x)]$

Product Rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Quotient Rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Higher derivatives

For $y = f(x)$,
- first derivative: $\frac{dy}{dx}$ or $f'(x)$
- second derivative: $\frac{d^2y}{dx^2}$ or $f''(x)$

Increasing/decreasing functions

- $\frac{dy}{dx} > 0 \Leftrightarrow y = f(x)$ is increasing
- $\frac{dy}{dx} < 0 \Leftrightarrow y = f(x)$ is decreasing

11.1 Rules of differentiation (Power Rule, Constant Multiple Rule, and Sum or Difference Rule)

Objectives Checklist

- Use the notations $\frac{dy}{dx}$, $f'(x)$ and $\frac{d}{dx}[f(x)]$ to denote a derivative
- Apply the Power Rule, Constant Multiple Rule, and Sum or Difference Rule to differentiate power functions

Notes and Worked Examples

The derivative of a function $y = f(x)$ is denoted by the **notations** $\frac{dy}{dx}$, $f'(x)$ and $\frac{d}{dx}[f(x)]$.
The gradient of the tangent to the curve $y = f(x)$ at a particular point is equal to the value of the derivative at that point.

To find the derivative of a power function x^n, where n is a real number, the **Power Rule** states that:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

In particular, the derivative of a constant k is 0:

$$\frac{d}{dx}(k) = 0$$

To find the derivative of a function $kf(x)$, where k is a constant and $f(x)$ is a function, the **Constant Multiple Rule** states that:

$$\frac{d}{dx}[kf(x)] = k\frac{d}{dx}[f(x)]$$

To find the derivative of the sum and difference of two functions $f(x)$ and $g(x)$, the **Sum Rule** and **Difference Rule** state that:

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

WORKED EXAMPLE 1 **Applying the Power Rule**

Differentiate each of the following with respect to x.

(a) $y = x^{20}$

(b) $f(x) = \frac{1}{x}$

(c) $\sqrt[3]{x}$

SOLUTION

(a) $y = x^{20}$

$\frac{dy}{dx} = \mathbf{20x^{19}}$

(b) $f(x) = \frac{1}{x}$

$= x^{-1}$

$f'(x) = -x^{-2}$

$= -\frac{\mathbf{1}}{\mathbf{x^2}}$

(c) $\frac{d}{dx}(\sqrt[3]{x}) = \frac{d}{dx}\left(x^{\frac{1}{3}}\right)$

$= \frac{1}{3}x^{-\frac{2}{3}}$

$= \frac{\mathbf{1}}{\mathbf{3\sqrt[3]{x^2}}} \leftarrow a^{-n} = \frac{1}{a^n}$

TRY TUTORIAL 11.1: Question 1

WORKED EXAMPLE 2 **Applying the Scalar Multiple Rule**

Differentiate each of the following with respect to x.

(a) $y = 9x$

(b) $f(x) = -\frac{4}{x^2}$

(c) $\frac{3}{2}\sqrt{x}$

SOLUTION

(a) $y = 9x$

$\frac{dy}{dx} = 9(1x^0)$

$= \mathbf{9}$

(b) $f(x) = -\frac{4}{x^2}$

$= -4x^{-2}$

$f'(x) = -4(-2x^{-3})$

$= \frac{\mathbf{8}}{\mathbf{x^3}}$

(c) $\frac{d}{dx}\left(\frac{3}{2}\sqrt{x}\right) = \frac{d}{dx}\left(\frac{3}{2}x^{\frac{1}{2}}\right)$

$= \frac{3}{2}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$

$= \frac{\mathbf{3}}{\mathbf{4\sqrt{x}}}$

TRY TUTORIAL 11.1: Question 2

WORKED EXAMPLE 3 Applying the Sum Rule or Difference Rule

Differentiate $8x^3+7-\frac{1}{\sqrt[3]{x}}$ with respect to x.

SOLUTION

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(8x^3+7-\frac{1}{\sqrt[3]{x}}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(8x^3+7-x^{-\frac{1}{3}}\right)$$

$$= 8(3x^2)+0-\left(-\frac{1}{3}x^{-\frac{4}{3}}\right)$$

$$= \mathbf{24x^2+\frac{1}{3\sqrt[3]{x^4}}} \longleftarrow a^{-n} = \frac{1}{a^n}$$

TRY TUTORIAL 11.1: Questions 3–5

WORKED EXAMPLE 4 Finding the gradient of a curve

Find the gradient of the curve $y = \frac{4x^2-7}{2x}$, where $x \neq 0$, at the point (–1, 1.5).

SOLUTION

$$y = \frac{4x^2-7}{2x}$$

$$= 2x-\frac{7}{2}x^{-1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2-\frac{7}{2}(-x^{-2})$$

$$= 2+\frac{7}{2x^2}$$

When $x = -1$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2+\frac{7}{2(-1)^2}$$

$$= 5.5$$

∴ Gradient of the curve at (–1, 1.5) is **5.5**

TRY TUTORIAL 11.1: Questions 6-9

NAME: ______________________ CLASS: __________ DATE: __________

TUTORIAL 11.1

1. Differentiate each of the following with respect to x.
   (a) $y = x^8$
   (b) $f(x) = \sqrt{x}$
   (c) $x^{-2.5}$
   (d) $\frac{1}{\sqrt[3]{x}}$

2. Differentiate each of the following with respect to x.
   (a) $y = 6x^2$
   (b) $f(x) = -\frac{9}{x}$
   (c) $\frac{7}{8}x\sqrt{x}$
   (d) $\tan\frac{\pi}{4}$

3. Differentiate each of the following with respect to x.
   (a) $3x^4 - 5 + \frac{2}{x}$
   (b) $\frac{8-x^2}{x^2}$

4. Given that $f(x) = (x+1)(\sqrt{x}+1)(\sqrt{x}-1)$, find an expression for $f'(x)$.

5. It is given that $y = \frac{1}{5}x(2x-3)$.
   (i) Find the value of $\frac{dy}{dx}$ when $x = 1$.
   (ii) Explain the significance of your answer to part (i).

6. Find the gradient(s) of the tangent(s) to the curve at the given point.
   (a) $y = 8x - x^2 + 3$, when $x = -\frac{1}{2}$
   (b) $y = \frac{2}{\sqrt[4]{x}}$, when $y = 1$
   (c) $y = (6x + 1)(x - 2)$, when the curve intersects the y-axis
   (d) $y = \frac{(x-5)^2}{x}$, when the curve intersects the x-axis

7. The gradient of the curve $y = ax + \frac{b}{x^2}$ at $x = -\frac{1}{3}$ is 2. Give an example of the values of the constants a and b.

8. The tangent to the curve $y = \frac{a}{\sqrt{x}} - bx + 9$ at (4, 10) is parallel to the line $5x + 8y + 6 = 0$. Find the value of a and of b.

9. A quadratic curve passes through the points (2, −1) and (−4, 29). The gradient of the tangent to the curve at $x = 3$ is parallel to the line $y + x = 29$. Find the equation of the curve.


IDEAS

11.2 Rules of differentiation (Chain Rule)

Objectives Checklist

- Apply the Chain Rule to differentiate functions

Notes and Worked Examples

To find the derivative of a function $[f(x)]^n$, where n is a constant and $f(x)$ is a function, the **Chain Rule** states that:

$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}\frac{d}{dx}[f(x)]$$

One way to remember the Chain Rule is:

Step 1: Bring down the power.

Step 2: Reduce the power by 1.

Step 3: Differentiate the "inside".

WORKED EXAMPLE 1 Applying the Chain Rule

Differentiate each of the following with respect to x.

(a) $y = (4x^2 - 3)^9$

(b) $f(x) = \frac{5}{2(1-6x)^8}$

SOLUTION

(a) $y = (4x^2 - 3)^9$

$$\frac{dy}{dx} = 9(4x^2 - 3)^8(8x)$$
$$= \mathbf{72x(4x^2 - 3)^8}$$

(b) $f(x) = \frac{5}{2(1-6x)^8}$

$$= \frac{5}{2}(1-6x)^{-8}$$

$$f'(x) = \frac{5}{2}(-8)(1-6x)^{-9}(-6)$$
$$= \mathbf{\frac{120}{(1-6x)^9}}$$

TRY TUTORIAL 11.2: Questions 1–3

WORKED EXAMPLE 2 **Finding the values of unknowns involving the gradient of a curve**

The gradient of the curve $y = 4\sqrt{5x - h}$, where $x \geqslant \frac{h}{5}$, at the point (1, k) is 5. Find the values of the constants h and k.

SOLUTION

$$y = 4\sqrt{5x-h} \qquad \text{— (1)}$$

$$= 4(5x-h)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 4\left(\frac{1}{2}\right)(5x-h)^{-\frac{1}{2}}(5)$$

$$= \frac{10}{\sqrt{5x-h}} \qquad \text{— (2)}$$

Substitute $x = 1$, $\frac{dy}{dx} = 5$ into (2):

$$5 = \frac{10}{\sqrt{5-h}}$$

$$\sqrt{5x - h} = 2$$

$$5 - h = 4$$

$$h = 1$$

Substitute $x = 1$, $h = 1$, $y = k$ into (1):

$$k = 4\sqrt{5-1}$$

$$= 8$$

$\therefore$ **$h = 1$, $k = 8$**

TRY TUTORIAL 11.2: Questions 4–6

NAME: ______________________ CLASS: __________ DATE: __________

TUTORIAL **11.2**

1. Differentiate each of the following with respect to x.
   (a) $(6x+1)^5$
   (b) $(2-9x)^8$
   (c) $\left(5+\frac{1}{4}x^2\right)^{12}$
   (d) $\frac{1}{3x-10}$
   (e) $2(3x^6-x-8)^6$
   (f) $\frac{7}{(9-2x^2)^3}$
   (g) $-\frac{10}{\sqrt[3]{7+5x}}$
   (h) $\frac{4}{5}\left(\sqrt{x^3}-\frac{1}{x}\right)^9$

2. Given that $y=\frac{2}{\sqrt{8-x}}$, show that $8\frac{\mathrm{d}y}{\mathrm{d}x}-y^3=0$.

3. (i) Express $\frac{3x+2}{(4x+1)^2}$ in partial fractions.
   (ii) Hence, find the derivative of $\frac{3x+2}{(4x+1)^2}$.

4. Find the gradient(s) of the tangent(s) to the curve at the given point.
   (a) $y=(7x-4)^3$, when $x=\frac{1}{2}$
   (b) $y=\sqrt{9+2x}$, when $y=1$
   (c) $y=(1-3x)\sqrt{1-3x}$, when the curve intersects the y-axis
   (d) $y=\frac{(5x+2)^6}{\sqrt[3]{x^{18}}}$, when the curve intersects the x-axis

5. The gradient of the curve $y=\left(a\sqrt{x}+b\right)^6+1$ at the point $(1, 2)$ is -9. Find the values of a and b.

6. The tangent to the curve $y=\frac{9}{1-x}$ at the point (p, q), where $p<0$, is parallel to the line $y-3x=1$. Show that the distance between (p, q) and the origin is $\sqrt{31-2\sqrt{3}}$ units.

11.3 Rules of differentiation (Product Rule)

Objectives Checklist

- Apply the Product Rule to differentiate functions

Notes and Worked Examples

To find the derivative of a product of two functions u and v, such that both u and v are functions of x, the **Product Rule** states that:

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

One way to remember the Product Rule is:

Step 1: Copy the first term and multiply it to the derivative of the second term.

Step 2: Copy the second term and multiply it to the derivative of the first term.

Step 3: Add the terms in Steps 1 and 2.

WORKED EXAMPLE 1 Applying the Product Rule

Differentiate each of the following with respect to x.

(a) $y = (x^3 - 2)(6 - x)^4$ **(b)** $f(x) = x^2\sqrt{4x+1}$

SOLUTION

(a) $y = (x^3 - 2)(6 - x)^4$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (x^3 - 2) \times 4(6 - x)^3(-1) + (6 - x)^4 \times 3x^2 \\ &= -4(x^3 - 2)(6 - x)^3 + 3x^2(6 - x)^4 \\ &= (6 - x)^3[-4(x^3 - 2) + 3x^2(6 - x)] \quad \leftarrow \text{Factorise the expression.} \\ &= (6 - x)^3[-4x^3 + 8 + 18x^2 - 3x^3] \\ &= \mathbf{(6 - x)^3(8 + 18x^2 - 7x^3)}\end{aligned}$$

(b) $f(x) = x^2\sqrt{4x+1}$

$= x^2(4x+1)^{\frac{1}{2}}$

$f'(x) = x^2 \times \frac{1}{2}(4x+1)^{-\frac{1}{2}}(4) + (4x+1)^{\frac{1}{2}} \times 2x$

$= \frac{2x^2}{\sqrt{4x+1}} + 2x\sqrt{4x+1}$

$= \frac{2x^2 + 2x(4x+1)}{\sqrt{4x+1}}$ ⟵ $\sqrt{4x+1} \times \sqrt{4x+1} = 4x + 1$

$= \frac{2x^2 + 8x^2 + 2x}{\sqrt{4x+1}}$

$= \mathbf{\frac{10x^2 + 2x}{\sqrt{4x+1}}}$

TRY TUTORIAL 11.3: Questions 1–4

WORKED EXAMPLE 2 Finding the conditions for the gradient of a curve to be zero

The equation of a curve is $y = (6x + a)(b - x)^2$. The tangent to the curve at $x = 1$ is parallel to the line $y = 12x - 7$. Find a possible pair of the values of a and b.

SOLUTION

$y = (6x + a)(b - x)^2$

$\frac{dy}{dx} = (6x + a) \times 2(b - x)(-1) + (b - x)^2 \times 6$

$= -2(6x + a)(b - x) + 6(b - x)^2$

$= 2(b - x)[-(6x + a) + 3(b - x)]$

$= 2(b - x)(-6x - a + 3b - 3x)$

$= 2(b - x)(3b - a - 9x)$ — (1)

Substitute $x = 1$, $\frac{dy}{dx} = 12$ into (1):

$2(b - 1)(3b - a - 9) = 12$

$(b - 1)(3b - a - 9) = 6$

Let $b = 2$: $6 - a - 9 = 6$

$a = -9$

∴ A possible pair of the values is $\mathbf{a = -9}$ and $\mathbf{b = 2}$.

TRY TUTORIAL 11.3: Questions 5, 6

NAME: ______________________ CLASS: __________ DATE: __________

TUTORIAL 11.3

1. Differentiate each of the following with respect to x.
   (a) $x(5x + 2)^8$
   (b) $(6x - 1)(10 - 3x)^4$
   (c) $(4x^2 - x + 7)(2x + 1)^5$
   (d) $(2x + 3)(2x - 3)(2x^2 + 3)^9$

2. Differentiate each of the following with respect to x.
   (a) $\sqrt{x}(x+1)^6$
   (b) $8x\sqrt{1-4x}$
   (c) $(3x+1)\sqrt{7x-2}$
   (d) $\left(\frac{1}{2}\sqrt{x}+5\right)\sqrt{9-x^2}$

3. By first expressing each expression as a product of two factors, differentiate each of the following with respect to x.
   (a) $x(6x + 1)(x - 3)^8$
   (b) $\frac{4}{x}(x-8)(10x^2+7)^5$

4. Given that $f(x) = (\sqrt{x}-1)\sqrt{x+4}$, show that $f'(x)$ can be expressed in the form $\frac{2x+a\sqrt{x}+4}{2\sqrt{x^2+4x^b}}$, where a and b are constants to be found.

5. The gradient of the curve $y = a(3x-1)^3\sqrt{x+a}$ at the point (0, −8) is m. Find the value of a and of m.

6. The equation of a curve is $y = (x^2 + 5)(2 - px)^7$, where p is a non-zero constant.
   (i) Find an expression for $\frac{dy}{dx}$.
   (ii) Hence, show that the gradient of the tangent to the curve can be 0 for values of p except $p < -\frac{2}{\sqrt{315}}$ and $p > \frac{2}{\sqrt{315}}$.

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TUTORIAL 11.3

1. Differentiate each of the following with respect to x.

   (a) $x(5x+2)^8$
   (b) $(6x-1)(10-3x)^4$
   (c) $(4x^2-x+7)(2x+1)^5$
   (d) $(2x+3)(2x-3)(2x^2+3)^9$

2. Differentiate each of the following with respect to x.

   (a) $\sqrt{x}(x+1)^6$
   (b) $8x\sqrt{1-4x}$
   (c) $(3x+1)\sqrt{7x-2}$
   (d) $\left(\frac{1}{2}\sqrt{x}+5\right)\sqrt{9-x^2}$

3. By first expressing each expression as a product of two factors, differentiate each of the following with respect to x.

   (a) $x(6x+1)(x-3)^8$
   (b) $\frac{4}{x}(x-8)(10x^2+7)^5$

4. Given that $f(x) = (\sqrt{x}-1)\sqrt{x+4}$, show that $f'(x)$ can be expressed in the form $\frac{2x+a\sqrt{x}+4}{2\sqrt{x^2+4x^b}}$, where a and b are constants to be found.

5. The gradient of the curve $y = a(3x-1)^3\sqrt{x+a}$ at the point $(0, -8)$ is m. Find the value of a and of m.

6. The equation of a curve is $y = (x^2+5)(2-px)^7$, where p is a non-zero constant.

   (i) Find an expression for $\frac{dy}{dx}$.

   (ii) Hence, show that the gradient of the tangent to the curve can be 0 for values of p except $p < -\frac{2}{\sqrt{315}}$ and $p > \frac{2}{\sqrt{315}}$.

11.4 Rules of differentiation (Quotient Rule)

Objectives Checklist

- Apply the Quotient Rule to differentiate functions

Notes and Worked Examples

To find the derivative of a quotient of two functions u and v, such that both u and v are functions of x, the **Quotient Rule** states that:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

One way to remember the Quotient Rule is:

Step 1: Square the denominator.

Step 2: Copy the denominator and multiply it to the derivative of the numerator.

Step 3: Copy the numerator and multiply it to the derivative of the denominator.

Step 4: Subtract the term in Step 3 from the term in Step 2 to form the numerator.

WORKED EXAMPLE 1 Applying the Quotient Rule

Differentiate each of the following with respect to x.

(a) $y = \frac{x^2+4}{7-x}$

(b) $f(x) = \frac{3x}{\sqrt{8x-1}}$

SOLUTION

(a) $y = \frac{x^2+4}{7-x}$

$$\frac{dy}{dx} = \frac{(7-x)\times 2x-(x^2+4)\times(-1)}{(7-x)^2}$$

$$= \frac{14x-2x^2+x^2+4}{(7-x)^2}$$

$$= \mathbf{\frac{4+14x-x^2}{(7-x)^2}}$$

(b) $f(x) = \dfrac{3x}{\sqrt{8x-1}}$

$= \dfrac{3x}{(8x-1)^{\frac{1}{2}}}$

$f'(x) = \dfrac{(8x-1)^{\frac{1}{2}} \times 3 - 3x \times \frac{1}{2}(8x-1)^{-\frac{1}{2}}(8)}{8x-1}$

$= \dfrac{3\sqrt{8x-1} - \dfrac{12x}{\sqrt{8x-1}}}{8x-1}$

$= \dfrac{3(8x-1)-12x}{\sqrt{(8x-1)^3}}$ ⟵ Multiply the numerator and denominator by $\sqrt{8x-1}$.

$= \dfrac{24x-3-12x}{\sqrt{(8x-1)^3}}$

$= \dfrac{12x-3}{\sqrt{(8x-1)^3}}$

TRY TUTORIAL 11.4: Questions 1–4

WORKED EXAMPLE 2

Showing that the gradient of a curve is not parallel to the x-axis

Show that there is no point on the curve $y = \dfrac{x+1}{2-x}$ for which the tangent is parallel to the x-axis.

SOLUTION

$y = \dfrac{x+1}{2-x}$

$\dfrac{dy}{dx} = \dfrac{(2-x)\times 1-(x+1)\times(-1)}{(2-x)^2}$

$= \dfrac{2-x+x+1}{(2-x)^2}$

$= \dfrac{3}{(2-x)^2}$

Since $\dfrac{dy}{dx} \neq 0$ for all real values of x, $x \neq 2$, then there is no point on the curve for which the tangent is parallel to the x-axis. (shown)

The gradient of a tangent that is parallel to the x-axis is 0.

TRY TUTORIAL 11.4: Question 5

NAME: ______________________ CLASS: ____________ DATE: __________

TUTORIAL 11.4

1. Differentiate each of the following with respect to x.

   (a) $\frac{x}{10x-3}$ (b) $\frac{7x}{2+x^2}$

   (c) $\frac{8x-1}{4x+5}$ (d) $\frac{6x^3}{1-9x}$

2. Differentiate each of the following with respect to x.

   (a) $\frac{5x}{\sqrt{x+7}}$ (b) $\frac{\sqrt{x}}{7x-4}$

   (c) $\frac{3x^2}{1-6\sqrt{x}}$ (d) $\frac{2x-9}{\sqrt[3]{3+8x}}$

3. Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{4-x}{\sqrt{3x+2}}\right)$ can be expressed in the form $-\frac{ax+b}{2\sqrt{(3x+2)^3}}$, where a and b are integers to be found.

4. It is given that $y=\frac{hx+7}{hx^2-1}$, where h is a constant.

   (i) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

   (ii) Find the range of values of h for which the equation $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ has real roots.

5. (i) Factorise completely the cubic polynomial $6x^3-24x^2+x-4$.

   (ii) Express $\frac{97}{6x^3-24x^2+x-4}$ in partial fractions.

   (iii) Hence, find the gradient of the tangent to the curve $y=\frac{97}{6x^3-24x^2+x-4}$ at the point where the curve intersects the y-axis.

11.5 Higher derivatives

Objectives Checklist

- Find higher derivatives of functions

Notes and Worked Examples

The derivative of a function $y = \mathrm{f}(x)$ is denoted by the **notations** $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\mathrm{f}'(x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f}(x)]$. If these **first derivatives** are differentiated again, the resulting functions, known as the **second derivatives**, are $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, $\mathrm{f}''(x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f}(x)]\right\}$ respectively.

WORKED EXAMPLE 1 Finding higher derivatives of a function

Find the first and second derivatives of $\frac{6x^2+1}{2x}$ with respect to x.

SOLUTION

Let $y = \frac{6x^2+1}{2x} = 3x+\frac{1}{2}x^{-1}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3+\frac{1}{2}(-x^{-2})$$ ← Alternatively, apply the Quotient Rule.

$$= 3-\frac{1}{2}x^{-2}$$

$$= 3-\frac{1}{2x^2}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0-\frac{1}{2}(-2x^{-3})$$

$$= \frac{1}{x^3}$$

TRY TUTORIAL 11.5: Questions 1–4

WORKED EXAMPLE 2 **Determining whether $\frac{d^2y}{dx^2}$ is equal to $\left(\frac{dy}{dx}\right)^2$**

Given that $y = 4x^3 - 1$, determine whether $\frac{d^2y}{dx^2}$ is equal to $\left(\frac{dy}{dx}\right)^2$.

SOLUTION

$y = 4x^3 - 1$

$\frac{dy}{dx} = 4(3x^2) - 0$

$= 12x^2$

$\frac{d^2y}{dx^2} = 12(2x)$

$= 24x$

Since $\left(\frac{dy}{dx}\right)^2 = (12x^2)^2 = 144x^4 \neq 24x$, then $\frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2$.

TRY TUTORIAL 11.5: Questions 5–7

NAME: ______________________ CLASS: ____________ DATE: __________

TUTORIAL 11.5

1. Find the first and second derivatives of each of the following with respect to x.

   (a) $y = \frac{1}{4}x + 9$

   (b) $y = 7x^3 - \frac{2}{x}$

   (c) $y = \frac{1}{\sqrt{x+8}}$

   (d) $y = (5 + 3x^2)^8$

   (e) $y = x(6x - 1)^4$

   (f) $y = \frac{x^2}{x+4}$

2. It is given that $f(x) = \left(\frac{1}{6}x^4 + 11\right)^9$.

   (i) Find $f'(x)$.

   (ii) Show that $f''(x) = x^2\left(\frac{1}{6}x^4 + 11\right)^7(35x^4 + 198)$.

3. If $y = \frac{x-a}{x-b}$, where a and b are constants, show that $(b-x)\frac{d^2y}{dx^2} = k\frac{dy}{dx}$, where k is a constant to be found.

4. Two functions $f(x)$ and $g(x)$ are such that $f(x) = \frac{3}{4}ax^2 + 5\sqrt{x} - 77$ and $g'(x) = \frac{b}{1-x}$, where a and b are proper fractions. Give an example of the values of a and b such that $8f''(9) + 48g''(5) = 3$.

5. Given that $y = \sqrt{x^2+1}$, determine whether $\frac{d^2y}{dx^2}$ is equal to $\left(\frac{dy}{dx}\right)^2$.

6. It is given that $\frac{y}{x^2-3} = x^4 + 3x^2 + 9$.

   (i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

   (ii) Show that $\frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2$.

   (iii) Find a relationship between $\frac{d^2y}{dx^2}$ and y.

7. Given the functions $f(x)$ and $g(x)$, which of the following statements are true? Explain your answer.

   (a) If $f(x) = g(x)$, then $f'(k) = g'(k)$, where k is a constant.

   (b) $f''(x)$ can be equal to $g''(x)$ even if $f(x) \neq g(x)$.

11.6 Increasing and decreasing functions

Objectives Checklist

- Determine whether a function is increasing or decreasing

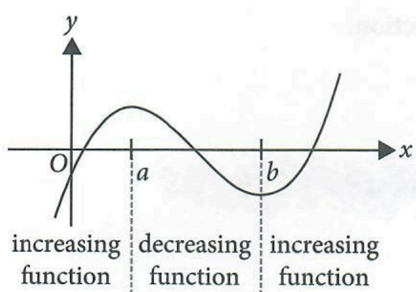
Notes and Worked Examples

The diagram shows part of a cubic curve $y = \mathrm{f}(x)$.

When $x < a$, as x increases, y increases, so $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.

When $a < x < b$, as x increases, y decreases, so $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$.

When $x > b$, as x increases, y increases, so $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.



To determine whether a function $y = \mathrm{f}(x)$ is increasing or decreasing:

- $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ in a given interval $\Leftrightarrow$ $y = \mathrm{f}(x)$ is increasing in that interval
- $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ in a given interval $\Leftrightarrow$ $y = \mathrm{f}(x)$ is decreasing in that interval

WORKED EXAMPLE 1

Finding the range of values of x given that y is increasing (or decreasing)

(i) Given that $y = 4x^3 + 4x^2 - 4x - 9$, find the range of values of x for which y is an increasing function.

(ii) Write down the range of values of x for which y is a decreasing function.

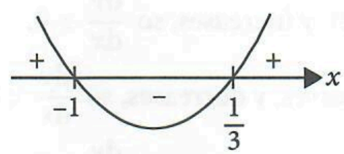
SOLUTION

(i) $y = 4x^3 + 4x^2 - 4x - 9$

$$\frac{dy}{dx} = 12x^2 + 8x - 4$$
$$= 4(3x^2 + 2x - 1)$$
$$= 4(3x - 1)(x + 1)$$

Since y is an increasing function, $\frac{dy}{dx} > 0$.

$$4(3x - 1)(x + 1) > 0$$
$$(3x - 1)(x + 1) > 0$$
$$x < -1 \text{ or } x > \frac{1}{3}$$



(ii) For y to be a decreasing function,

$-1 < x < \frac{1}{3}$.

TRY TUTORIAL 11.6: Questions 1–6

WORKED EXAMPLE 2 Showing that a function is decreasing in a particular interval

Show that the function $\frac{x^2+16}{x}$, for $x \neq 0$, is decreasing in the interval $-4 < x < 4$, $x \neq 0$.

SOLUTION

Let $y = \frac{x^2+16}{x} = x + 16x^{-1}$.

$$\frac{dy}{dx} = 1 + 16(-x^{-2})$$

$$= 1 - \frac{16}{x^2}$$

For the function to be decreasing, $\frac{dy}{dx} < 0$.

$$1 - \frac{16}{x^2} < 0$$

$$\frac{x^2 - 16}{x^2} < 0$$

$$x^2 - 16 < 0 \longleftarrow x^2 > 0 \text{ for all real values of } x, x \neq 0$$

$$(x+4)(x-4) < 0$$

$$-4 < x < 4$$

∴ The function is decreasing in the interval $-4 < x < 4$, $x \neq 0$. (shown)

TRY TUTORIAL 11.6: Questions 7, 8

IDEAS

NAME: ______ CLASS: ______ DATE: ______

TUTORIAL 11.6

1. The function f is defined, for all real values of x, by

$$f(x) = 4x^2(2 - 3x).$$

Find the range of values of x for which f is an increasing function.

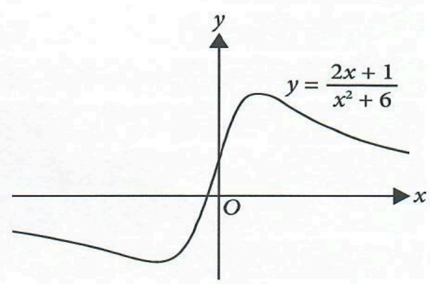
2. Find the set of values of t for which $f(t) = t^4 - 1$ is a decreasing function.

3. Find the range of values of x for which

(a) $f(x) = 2x^3 + 5x^2 - 16x - 49$ is an increasing function,

(b) $f'(x) = 25 - x^2$ is a decreasing function.

4. The diagram shows part of the graph of $y = \frac{2x+1}{x^2+6}$. Find the values of x for which y is increasing.



5. The equation of a curve is $y = x^3 + 3x^2 - 45x + k$, where k is a constant.

(i) Find the set of values of x for which y is increasing.

(ii) Find the possible values of k for which the x-axis is a tangent to the curve.

6. It is given that $y = x^3 - ax^2 + bx - 7$, where a and b are integers. The only values of x for which y is a decreasing function are those values for which $-1 < x < 5$. Find the value of a and of b.

7. Show that the function $\frac{x}{\sqrt{5x-2}}$, for $x > \frac{2}{5}$, is decreasing in the interval $\frac{2}{5} < x < \frac{4}{5}$.

8. **Value of a painting.** Jacob recorded the market value, \$$V$, of a painting regularly over a period of t years and plotted a graph of V against t. Jacob believes that the model $V = \frac{20\,000(1+t)}{1+0.5t^2}$ is a good fit for the data.

(i) State the initial value of the painting.

(ii) Jacob regrets not selling the painting in the first year, and he thinks that the value of the painting will continue to decrease. Suggest how he might have used the derivative of a function to draw this conclusion.

IDEAS

NAME: ______________________ CLASS: ____________ DATE: __________

Quick Test 11

[Total marks: 20]

1. Differentiate each of the following with respect to x.

   (a) $\frac{\sqrt{x}}{2} - \frac{\sqrt{2}}{x} + \pi$ [3]

   (b) $(5x^3 + 1)(6 - x^2)$ [2]

2. Find $\frac{d}{dx}\left[\frac{8}{(4-3x)^6}\right]$

   (a) by writing the function in the form $8(4 - 3x)^n$, where n is a constant, and applying the Chain Rule, [2]

   (b) by using the Quotient Rule. [2]

3. Write down a possible function f(x) for which $f''(x) = 0.25x^2 + 4$. [2]

4. The equation of a curve is $y = \frac{1}{7}x\sqrt{x^2 - 9}$.

(i) State the range of values of x for which this curve exists. [1]

(ii) Find the acute angle, in radians, that the tangent to the curve $y = \frac{1}{7}x\sqrt{x^2 - 9}$ at the point where $x = -5$ makes with the x-axis. [6]

(iii) Determine whether $y = \frac{1}{7}x\sqrt{x^2 - 9}$ is always increasing as x increases from 3. [2]