CHAPTER 3 Surds

Chapter Summary

Indices — are linked to — **Surds**

Indices have properties:

- **Law 1:** $a^m \times a^n = a^{m+n}$
- **Law 2:** $\frac{a^m}{a^n} = a^{m-n}$
- **Law 3:** $(a^m)^n = a^{mn}$
- **Law 4:** $a^n \times b^n = (ab)^n$
- **Law 5:** $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$

Surds can be multiplied or divided:

- **Law 1:** $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- **Law 2:** $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
- **Law 3:** $\sqrt{a} \times \sqrt{a} = a$

Rationalise the denominator of a fraction by multiplying both the numerator and the denominator by the conjugate of the denominator.

Surds can be added or subtracted:

$m\sqrt{a} \pm n\sqrt{a} = (m \pm n)\sqrt{a}$

Equations involving surds

can be solved by

Equality of Surds:

$m+n\sqrt{a} = p+q\sqrt{a} \Leftrightarrow m = p$ and $n = q$

Taking the square on both sides:

$\sqrt{x} = k \Rightarrow x = k^2$

3.1 Simplifying expressions involving surds

Objectives Checklist

- Simplify expressions involving surds
- Rationalise the denominator of an expression containing a surd

Notes and Worked Examples

A surd is an irrational root of a real number, e.g. $\sqrt{2}$ and $\sqrt[3]{5}$.

If $a > 0$ and $b > 0$, then the **Laws of Surds** state that:

Law 1: $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

Law 2: $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Law 3: $\sqrt{a} \times \sqrt{a} = a$

Other operations on surds include:

$$m\sqrt{a} \pm n\sqrt{a} = (m \pm n)\sqrt{a}$$

A fraction with a surd in the denominator, e.g. $\frac{6}{\sqrt{3}}$, can be **rationalised** by multiplying both the numerator and the denominator by $\sqrt{3}$, to get its **equivalent** form $\frac{6\sqrt{3}}{3} = 2\sqrt{3}$.

Since the product of a pair of conjugate surds is always a rational number, rationalise the denominator by multiplying both the numerator and the denominator by the **conjugate**:

Case	Fraction	Multiply by
1	$\frac{h}{\sqrt{a}}$	$\frac{\sqrt{a}}{\sqrt{a}}$
2	$\frac{h}{\sqrt{a}+b}$	$\frac{\sqrt{a}-b}{\sqrt{a}-b}$
3	$\frac{h}{\sqrt{a}+\sqrt{b}}$	$\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}$
4	$\frac{h}{p\sqrt{a}+q\sqrt{b}}$	$\frac{p\sqrt{a}-q\sqrt{b}}{p\sqrt{a}-q\sqrt{b}}$

WORKED EXAMPLE 1 **Simplifying surds**

Simplify each of the following.

(a) $(\sqrt{3})^2$

(b) $\sqrt{50}$

(c) $\sqrt{5}\times\sqrt{125}$

(d) $\dfrac{\sqrt{180}}{\sqrt{15}}$

SOLUTION

(a) $(\sqrt{3})^2 = \sqrt{3}\times\sqrt{3}$
$= 3$

(b) $\sqrt{50} = \sqrt{25\times 2}$
$= \sqrt{25}\times\sqrt{2}$
$= 5\sqrt{2}$

(c) $\sqrt{5}\times\sqrt{125} = \sqrt{5\times 125}$
$= \sqrt{5\times 5\times 5\times 5}$
$= \sqrt{(5\times 5)\times(5\times 5)}$
$= 25$

(d) $\dfrac{\sqrt{180}}{\sqrt{15}} = \sqrt{\dfrac{180}{15}}$
$= \sqrt{12}$
$= \sqrt{4\times 3}$
$= \sqrt{4}\times\sqrt{3}$
$= 2\sqrt{3}$

TRY TUTORIAL 3.1: Question 1

WORKED EXAMPLE 2 **Adding and subtracting surds**

Simplify each of the following.

(a) $8\sqrt{2}+3\sqrt{2}-7\sqrt{2}$

(b) $\sqrt{27}-\sqrt{243}+5\sqrt{3}$

SOLUTION

(a) $8\sqrt{2}+3\sqrt{2}-7\sqrt{2} = (8+3-7)\sqrt{2}$
$= 4\sqrt{2}$

(b) $\sqrt{27}-\sqrt{243}+5\sqrt{3} = \sqrt{9\times 3}-\sqrt{81\times 3}+5\sqrt{3}$
$= 3\sqrt{3}-9\sqrt{3}+5\sqrt{3}$
$= (3-9+5)\sqrt{3}$
$= -\sqrt{3}$

TRY TUTORIAL 3.1: Question 2

WORKED EXAMPLE 3 Multiplying surds

Simplify each of the following.

(a) $(5+3\sqrt{2})(4-2\sqrt{2})$

(b) $(10-\sqrt{7})^2$

(c) $(9+4\sqrt{5})(9-4\sqrt{5})$

(d) $(5\sqrt{6}+4\sqrt{3})^2$

SOLUTION

(a) $(5+3\sqrt{2})(4-2\sqrt{2})$

$= 20-10\sqrt{2}+12\sqrt{2}-6(\sqrt{2})^2$

$= 20-10\sqrt{2}+12\sqrt{2}-12$

$= \mathbf{8+2\sqrt{2}}$

(b) $(10-\sqrt{7})^2$

$= 10^2-2(10)(\sqrt{7})+(\sqrt{7})^2$

$= 100-20\sqrt{7}+7$

$= \mathbf{107-20\sqrt{7}}$

(c) $(9+4\sqrt{5})(9-4\sqrt{5})$

$= 9^2-(4\sqrt{5})^2$

$= 81-16(\sqrt{5})^2$

$= 81-80$

$= \mathbf{1}$

(d) $(5\sqrt{6}+4\sqrt{3})^2$

$= (5\sqrt{6})^2+2(5\sqrt{6})(4\sqrt{3})+(4\sqrt{3})^2$

$= 25(\sqrt{6})^2+40\sqrt{6\times 3}+16(\sqrt{3})^2$

$= 150+40\sqrt{9\times 2}+48$

$= 198+40(3\sqrt{2})$

$= \mathbf{198+120\sqrt{2}}$

TRY TUTORIAL 3.1: Questions 3–5

WORKED EXAMPLE 4 **Rationalising the denominator**

Simplify each of the following by rationalising the denominator.

(a) $\frac{10}{\sqrt{2}}$

(b) $\frac{12}{3+\sqrt{6}}$

(c) $\frac{\sqrt{3}}{\sqrt{5}-2}$

(d) $\frac{4\sqrt{3}-5\sqrt{2}}{4\sqrt{3}+5\sqrt{2}}$

SOLUTION

(a) $\frac{10}{\sqrt{2}} = \frac{10}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \mathbf{5\sqrt{2}}$

(b) $\frac{12}{3+\sqrt{6}} = \frac{12}{3+\sqrt{6}} \times \frac{3-\sqrt{6}}{3-\sqrt{6}}$

$= \frac{12(3-\sqrt{6})}{3^2-(\sqrt{6})^2}$

$= \frac{12(3-\sqrt{6})}{9-6}$

$= \frac{12(3-\sqrt{6})}{3}$

$= 4(3-\sqrt{6})$

$= \mathbf{12 - 4\sqrt{6}}$

(c) $\frac{\sqrt{3}}{\sqrt{5}-2} = \frac{\sqrt{3}}{\sqrt{5}-2} \times \frac{\sqrt{5}+2}{\sqrt{5}+2}$

$= \frac{\sqrt{15}+2\sqrt{3}}{(\sqrt{5})^2-2^2}$

$= \frac{\sqrt{15}+2\sqrt{3}}{5-4}$

$= \mathbf{\sqrt{15} + 2\sqrt{3}}$

(d) $\frac{4\sqrt{3}-5\sqrt{2}}{4\sqrt{3}+5\sqrt{2}} = \frac{4\sqrt{3}-5\sqrt{2}}{4\sqrt{3}+5\sqrt{2}} \times \frac{4\sqrt{3}-5\sqrt{2}}{4\sqrt{3}-5\sqrt{2}}$

$= \frac{(4\sqrt{3})^2-2(4\sqrt{3})(5\sqrt{2})+(5\sqrt{2})^2}{(4\sqrt{3})^2-(5\sqrt{2})^2}$

$= \frac{48-40\sqrt{6}+50}{48-50}$

$= \frac{98-40\sqrt{6}}{-2}$

$= \mathbf{20\sqrt{6} - 49}$

TRY TUTORIAL 3.1: Questions 6–12

NAME: ____________________ CLASS: __________ DATE: ______

TUTORIAL 3.1

1. Simplify each of the following.

(a) $(\sqrt{7})^2$

(b) $\sqrt{192}$

(c) $\sqrt{27}\times\sqrt{3}$

(d) $\dfrac{\sqrt{280}}{\sqrt{14}}$

2. Simplify each of the following.

(a) $5\sqrt{32}-4\sqrt{8}+\sqrt{128}$

(b) $\sqrt{500}+5\sqrt{5}-2\sqrt{125}$

3. Simplify each of the following.

(a) $(7-2\sqrt{3})(5+\sqrt{3})$

(b) $(4\sqrt{6}+9)^2$

(c) $(8+3\sqrt{2})(8-3\sqrt{2})$

(d) $(10\sqrt{5}-\sqrt{7})^2$

4. Express $(7+\sqrt{5})^2+(4-\sqrt{5})^2$ in the form $a+b\sqrt{5}$, stating the value of each of the integers a and b.

5. It is given that $p=\sqrt{5}-\sqrt{3}$.

(i) Without using a calculator, express p^2 in the form $\sqrt{a}-2\sqrt{b}$, where a and b are integers.

(ii) Hence, or otherwise, find the exact value of $3p^4-8p^2+1$.

6. Simplify each of the following by rationalising the denominator.

(a) $\dfrac{55}{\sqrt{11}}$

(b) $\dfrac{27}{\sqrt{10}-1}$

(c) $\dfrac{\sqrt{2}}{9+4\sqrt{2}}$

(d) $\dfrac{\sqrt{6}+\sqrt{2}}{2\sqrt{6}-5\sqrt{2}}$

7. Without using a calculator, express $\dfrac{11-\sqrt{7}}{3+\sqrt{7}}$ in the form $a+b\sqrt{7}$, where a and b are integers.

8. Express $\dfrac{9}{7+2\sqrt{3}}+\dfrac{4}{2\sqrt{3}-7}$ as a single fraction.

9. Express $(4-\sqrt{5})^2-\dfrac{8}{\sqrt{5}-2}$ in the form $a+b\sqrt{5}$, where a and b are integers to be found.

10. Given that $x=\dfrac{\sqrt{17}-4}{\sqrt{17}+4}$, show that $x+\dfrac{1}{x}=66$.

11. Without using a calculator, find the fractions, a and b, for which $\dfrac{\sqrt{21}+\sqrt{2}}{\sqrt{14}-\sqrt{3}}$ can be expressed as $a\sqrt{7}+b\sqrt{6}$.

12. **(i)** Without using a calculator, express $(5\sqrt{2}-3)(7+\sqrt{2})$ in the form $a\sqrt{2}+b$, where a and b are integers.

(ii) Hence, or otherwise, express $\dfrac{(5\sqrt{2}-3)(7+\sqrt{2})}{3+\sqrt{2}}$ in the form $p\sqrt{2}+q$, where p and q are rational numbers.

3.2 Solving equations involving surds

Objectives Checklist

- Solve equations involving surds

Notes and Worked Examples

In an equation involving surds, if a, m, n, p and q are rational numbers, the **Equality of Surds** states that:

$$m+n\sqrt{a} = p+q\sqrt{a} \Leftrightarrow m = p \text{ and } n = q$$

Solving an equation in x involving surds is the same as rearranging the terms to make x the subject of the formula. If we take the square on both sides of the equation, extraneous solutions may be introduced, so always check whether any solution has to be rejected.

WORKED EXAMPLE 1 Solving equations involving surds

Solve each of the following equations.

(a) $\sqrt{x+7} = 2$

(b) $\sqrt{5-4x} - 7 = 0$

(c) $\sqrt{x^2+8} = 3x$

(d) $\sqrt{3x-4} - 2\sqrt{x-1} = 0$

SOLUTION

(a)

$$\sqrt{x+7} = 2$$
$$\left(\sqrt{x+7}\right)^2 = 2^2$$
$$x + 7 = 4$$
$$x = -3$$

Check: LHS = $\sqrt{-3+7} = 2$ = RHS

(b)

$$\sqrt{5-4x} - 7 = 0$$
$$\sqrt{5-4x} = 7$$
$$\left(\sqrt{5-4x}\right)^2 = 7^2$$
$$5 - 4x = 49$$
$$4x = -44$$
$$x = -11$$

Check: LHS = $\sqrt{5-4(-11)} - 7 = 0$ = RHS

(c)

$$\sqrt{x^2+8} = 3x$$
$$\left(\sqrt{x^2+8}\right)^2 = (3x)^2$$
$$x^2 + 8 = 9x^2$$
$$8x^2 = 8$$
$$x^2 = 1$$
$$x = \pm 1$$

Check: When $x = 1$,

LHS = $\sqrt{1^2+8} = 3 =$ RHS

Check: When $x = -1$,

LHS = $\sqrt{(-1)^2+8} = 3 \neq$ RHS

$\therefore$ $\mathbf{x = 1}$

(d)

$$\sqrt{3x-4} - 2\sqrt{x-1} = 0$$
$$\sqrt{3x-4} = 2\sqrt{x-1}$$
$$\left(\sqrt{3x-4}\right)^2 = \left(2\sqrt{x-1}\right)^2$$
$$3x - 4 = 4(x-1)$$
$$3x - 4 = 4x - 4$$
$$x = 0$$

$\therefore$ The solution has **no real solutions.**

Check: When $x = 0$,

$\sqrt{3x-4} = \sqrt{-4}$ and $\sqrt{x-1} = \sqrt{-1}$

TRY TUTORIAL 3.2: Questions 1–4

WORKED EXAMPLE 2 Solving equations involving rationalising the denominator

Without using a calculator, solve the equation $2x - 5\sqrt{3} = x\sqrt{3} + 8$, giving your answer in the form $p + q\sqrt{3}$, where p and q are rational numbers.

SOLUTION

$$2x - 5\sqrt{3} = x\sqrt{3} + 8$$
$$2x - x\sqrt{3} = 8 + 5\sqrt{3}$$
$$x(2-\sqrt{3}) = 8 + 5\sqrt{3}$$
$$x = \frac{8+5\sqrt{3}}{2-\sqrt{3}}$$

← Rearrange the terms to make x the subject of the formula.

$$= \frac{8+5\sqrt{3}}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}$$
$$= \frac{(8+5\sqrt{3})(2+\sqrt{3})}{2^2 - (\sqrt{3})^2}$$
$$= \frac{16 + 8\sqrt{3} + 10\sqrt{3} + 15}{4-3}$$
$$= \frac{31+18\sqrt{3}}{1}$$
$$= \mathbf{31 + 18\sqrt{3}}$$

TRY TUTORIAL 3.2: Questions 5, 6

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WORKED EXAMPLE 3 **Solving equations involving comparison of terms**

Given that $(6+a\sqrt{7})(3+2\sqrt{7}) = 4+b\sqrt{7}$, where a and b are integers, find the value of a and of b.

SOLUTION

$$(6+a\sqrt{7})(3+2\sqrt{7}) = 4+b\sqrt{7}$$

$$18+12\sqrt{7}+3a\sqrt{7}+14a = 4+b\sqrt{7}$$

$$(14a+18)+(3a+12)\sqrt{7} = 4+b\sqrt{7}$$

Equating the rational terms,

$$14a + 18 = 4$$

$$14a = -14$$

$$a = -1$$

Equating the irrational terms,

$$3a + 12 = b$$

$$3(-1) + 12 = b$$

$$b = 9$$

$\therefore$ **$a = -1$, $b = 9$**

TRY TUTORIAL 3.2: Questions 7–11

WORKED EXAMPLE 4 **Solving equations involving mensuration**

The length of a rectangle is $(10+3\sqrt{3})$ cm and its area is $(75-14\sqrt{3})$ cm². Find its breadth in the form $(a+b\sqrt{3})$ cm, where a and b are integers.

SOLUTION

Let the breadth of the rectangle be y cm.

Area of rectangle = Length × breadth

$$75-14\sqrt{3} = (10+3\sqrt{3})y$$

$$y = \frac{75-14\sqrt{3}}{10+3\sqrt{3}}$$ ← Rearrange the terms to make y the subject of the formula.

$$= \frac{75-14\sqrt{3}}{10+3\sqrt{3}} \times \frac{10-3\sqrt{3}}{10-3\sqrt{3}}$$

$$= \frac{(75-14\sqrt{3})(10-3\sqrt{3})}{10^2-(3\sqrt{3})^2}$$

$$= \frac{750-225\sqrt{3}-140\sqrt{3}+126}{100-27}$$

$$= \frac{876-365\sqrt{3}}{73}$$

$$= 12-5\sqrt{3}$$

∴ The breadth of the rectangle is $(12-5\sqrt{3})$ **cm.**

TRY TUTORIAL 3.2: Questions 12–14

NAME: ______________________ CLASS: __________ DATE: __________

TUTORIAL 3.2

1. Solve each of the following equations.

(a) $\sqrt{x-6}=5$ **(b)** $\sqrt{10-x}-3=0$

(c) $\sqrt{2x^2+28}=3x$ **(d)** $2\sqrt{5x+9}-3\sqrt{4x-7}=0$

(e) $\sqrt{x-15}=\dfrac{4}{\sqrt{x}}$ **(f)** $\sqrt{25-\sqrt{2x+1}}=4$

2. Find the values of x which satisfy the equation $\dfrac{x}{\sqrt{29}+2}=\dfrac{\sqrt{29}-2}{x}$.

3. A student wrote the following to solve the equation $\sqrt{5x+7}+\sqrt{1-4x}=0$.

$$\sqrt{5x+7}+\sqrt{1-4x}=0$$
$$\sqrt{5x+7}=-\sqrt{1-4x}$$

Squaring both sides of the equation,

$$5x+7=1-4x$$
$$9x=-6$$
$$x=-\frac{2}{3}$$

What misconception has the student made? Explain why the equation has no real solutions.

4. Solve the equation $\sqrt{2x+4}+\sqrt{2x-3}=\sqrt{8x+1}$.

5. Solve the equation $x\sqrt{2}-x=10$.

6. Without using a calculator, solve the equation $4x+9\sqrt{5}=x\sqrt{5}-7$, giving your answer in the form $p+q\sqrt{5}$, where p and q are rational numbers.

7. Find the values of the constants a and b for which $a+b\sqrt{7}=\left(\sqrt{7}-2\right)^2-5\sqrt{7}$.

8. Given that $4+2\sqrt{3}$ is a root of the equation $x^2+ax+b=0$, where a and b are integers, find the value of a and of b.

9. Without using a calculator, find the values of the integers a and b such that $\dfrac{a+b\sqrt{2}}{5-4\sqrt{2}}=\dfrac{5-4\sqrt{2}}{3+2\sqrt{2}}$.

10. **(i)** Write down the expansion of $\left(\sqrt{a}+\sqrt{b}\right)^2$.

(ii) Hence, find the positive square root of $69+16\sqrt{5}$ in the form $c+\sqrt{d}$, where c and d are integers.

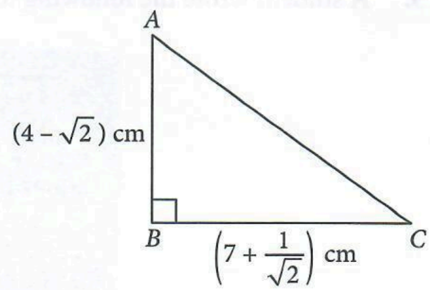
11. **(i)** Expand and simplify $\left(\sqrt[3]{3}+\sqrt[3]{2}\right)\left(\sqrt[3]{9}-\sqrt[3]{6}+\sqrt[3]{4}\right)$.

(ii) Hence, find the values of the rational numbers a and b in the equation

$$\frac{\sqrt[3]{6}-\sqrt[3]{4}}{\sqrt[3]{9}-\sqrt[3]{6}+\sqrt[3]{4}} = a\sqrt[3]{18}+b.$$

12. The perimeter of an equilateral triangle is $12\left(\sqrt{3}+2\right)$ cm. Given that $\sin 60° = \frac{\sqrt{3}}{2}$, show that the area of the triangle can be expressed as $\left(a+b\sqrt{3}\right)$ cm², where a and b are integers.

13. In the triangle ABC, $AB = \left(4-\sqrt{2}\right)$ cm, $BC = \left(7+\frac{1}{\sqrt{2}}\right)$ cm and angle $ABC = 90°$.



(i) Find the area of the triangle in the form $\left(a+b\sqrt{2}\right)$ cm², where a and b are rational numbers.

(ii) Find an expression for AC^2 in the form $\left(c+d\sqrt{2}\right)$ cm², where c and d are rational numbers.

14. A rectangular block has a square base of side $\left(\sqrt{15}+\sqrt{3}\right)$ cm and a height of h cm. The volume of the rectangular block is $\left(120+24\sqrt{5}\right)$ cm³. Without using a calculator, show that h can be expressed as $a+b\sqrt{5}$, where a and b are integers.

NAME: ____________________ **CLASS:** __________ **DATE:** __________

Quick Test 3

[Total marks: 20]

1. Without using a calculator, simplify $\frac{7}{\sqrt{32}} - \frac{\sqrt{128}}{\sqrt{8}} + \frac{10}{\sqrt{512}}$. [3]

2. Given that $p = \sqrt{7} - 2$, express $\frac{p^2 - 4}{p + 4}$ in the form $a\sqrt{7} + b$, where a and b are integers. [4]

3. Solve the equation $3x + \sqrt{4 - 5x} = 0$. [3]

4. Without using a calculator, find the values of the integers a and b such that $\frac{5-\sqrt{3}}{a+b\sqrt{3}} = \frac{2+\sqrt{3}}{5-\sqrt{3}}$. [4]

5. A rectangle of length $(9+\sqrt{12})$ cm has an area of $(48+\sqrt{27})$ cm^2.

   (a) Without using a calculator, find the breadth of the rectangle, in cm, in the form $(a+b\sqrt{3})$, where a and b are integers. [3]

   (b) The rectangle is the base of a cuboid with a height of $(6\sqrt{3}+5)$ cm. Find the volume of the cuboid, in cm^3, in the form $(c+d\sqrt{3})$, where c and d are integers. [3]