CHAPTER 4 Polynomials, Cubic Equations and Partial Fractions

Chapter Summary

Polynomials

- **Expressions**
  - **Remainder Theorem**
    If $f(x)$ is divided by $ax + b$, the remainder is $f\left(-\frac{b}{a}\right)$.
    - dividend = divisor × quotient + remainder
  - **Factor Theorem**
    $ax + b$ is a factor of $f(x)$ $\Leftrightarrow f\left(-\frac{b}{a}\right) = 0$.
- **Equations**
- **Identities**
  - $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

To factorise a cubic expression:

Step 1: Use the Factor Theorem to find a linear factor of $f(x)$, e.g. $x - p$.

Step 2: Write the cubic expression in the form $(x - p)(ax^2 + bx + c)$.

Step 3: Substitute suitable values of x or equate coefficients of like powers of x to find the values of a, b and c.

Step 4: Factorise $ax^2 + bx + c$ if possible.

→ To solve a cubic equation:

Step 5: Equate each factor of the cubic expression to 0.

Expressing an algebraic fraction in partial fractions

Improper fraction → **Step 1:** Use long division. → **Proper fraction**

Proper fraction

Step 2: Factorise the denominator Q(x) completely.

Step 3: Express $\frac{P(x)}{Q(x)}$ in partial fractions according to the different cases.

Step 4: Solve for the unknowns by substituting suitable values of x and/or comparing coefficients of like terms.

Partial fractions

- $Q(x) = ax + b \Rightarrow \frac{A}{ax+b}$
- $Q(x) = (ax + b)^2 \Rightarrow \frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
- $Q(x) = x^2 + c^2 \Rightarrow \frac{Ax+B}{x^2+c^2}$

4.1 Polynomials and identities

Objectives Checklist

- Add, subtract, multiply and divide polynomials
- Find the values of unknowns in identities
- State the Division Algorithm for Polynomials

Notes and Worked Examples

A polynomial in x is the sum of terms, each of the form ax^n, where n is a non-negative integer.

The degree of a polynomial is the highest power of x.

Type of polynomial	Example of polynomial	Degree
Linear expression	$10x + 9$	1
Quadratic expression	$3-\frac{1}{2}x^2$	2
Cubic expression	$4x^3 + 7x^2 - 1$	3

The value of a polynomial P(x) at $x = a$ is P(a).

If two polynomials P(x) and Q(x) are **equivalent**, then P(x) = Q(x) is true for all values of x. The equation is known as an **identity**.

To find the unknown coefficients in an identity, we can:

- Substitute suitable values of x (Method 1)
- Equate coefficients of like powers of x (Method 2)

The **Division Algorithm for Polynomials** states that:

dividend = divisor × quotient + remainder

The order of the remainder is always at least one degree less than that of the divisor.

WORKED EXAMPLE 1

Identifying polynomials

Determine whether each of the following is a polynomial. Explain your answer briefly.

(a) $3x^4 + 9x - 20$

(b) $2\sqrt{x} - 8$

(c) $\frac{4}{5}$

(d) $\frac{1}{x} + 1$

SOLUTION

(a) **Yes.** The terms in the expression are in the form ax^n, where each value of n is a non-negative integer.

(b) **No.** $2\sqrt{x} = 2x^{\frac{1}{2}}$ has a fractional exponent.

(c) **Yes.** $\frac{4}{5}$ is the constant term in a polynomial. ⟵ $n = 0$

(d) **No.** $\frac{1}{x} = x^{-1}$ has a negative exponent.

TRY TUTORIAL 4.1: Question 1

WORKED EXAMPLE 2

Adding and subtracting polynomials

It is given that $P(x) = 2x^3 + 6x^2 - 7x + 3$ and $Q(x) = 4x^2 + 7x - 5$.

(i) Find an expression for each of the following and state its degree.

(a) $P(x) + Q(x)$

(b) $4P(x) - 6Q(x)$

(ii) Show that $4P(-1) - 6Q(-1) = 104$.

SOLUTION

(i) (a) $P(x) + Q(x) = (2x^3 + 6x^2 - 7x + 3) + (4x^2 + 7x - 5)$

$= 2x^3 + 6x^2 - 7x + 3 + 4x^2 + 7x - 5$

$= \mathbf{2x^3 + 10x^2 - 2}$ ⟵ Group like terms.

Degree of $P(x) + Q(x)$ is **3**

(b) $4P(x) - 6Q(x) = 4(2x^3 + 6x^2 - 7x + 3) - 6(4x^2 + 7x - 5)$

$= 8x^3 + 24x^2 - 28x + 12 - 24x^2 - 42x + 30$

$= \mathbf{8x^3 - 70x + 42}$ ⟵ Group like terms.

Degree of $4P(x) - 6Q(x)$ is **3**

(ii) $4P(x) - 6Q(x) = 8x^3 - 70x + 42$

$$4P(-1) - 6Q(-1) = 8(-1)^3 - 70(-1) + 42$$
$$= 104 \text{ (shown)}$$

TRY TUTORIAL 4.1: Question 2

WORKED EXAMPLE 3 **Finding the degree of the sum or difference of polynomials**

Suggest a pair of polynomials $P(x)$ and $Q(x)$ for which the degree of $P(x) + Q(x)$ is greater than the degree of $P(x) - Q(x)$.

SOLUTION

A possible pair is $\mathbf{P(x) = 5x^3 + x^2 - 10}$ and $\mathbf{Q(x) = 5x^3 - 6x^2}$.

$$P(x) + Q(x) = (5x^3 + x^2 - 10) + (5x^3 - 6x^2)$$
$$= 5x^3 + x^2 - 10 + 5x^3 - 6x^2$$
$$= 10x^3 - 5x^2 - 10$$

Degree of $P(x) + Q(x)$ is 3

$$P(x) - Q(x) = (5x^3 + x^2 - 10) - (5x^3 - 6x^2)$$
$$= 5x^3 + x^2 - 10 - 5x^3 + 6x^2$$
$$= 7x^2 - 10$$

Degree of $P(x) - Q(x)$ is 2

TRY TUTORIAL 4.1: Question 3

WORKED EXAMPLE 4 Multiplying polynomials

Expand each of the following.

(a) $(8x + 3)(x^2 - 5x + 1)$ **(b)** $(4x^2 - x + 7)(2x^2 - 9)$

SOLUTION

(a) $(8x + 3)(x^2 - 5x + 1) = 8x(x^2 - 5x + 1) + 3(x^2 - 5x + 1)$

$= 8x^3 - 40x^2 + 8x + 3x^2 - 15x + 3$

$= 8x^3 - 37x^2 - 7x + 3$

(b) $(4x^2 - x + 7)(2x^2 - 9) = 4x^2(2x^2 - 9) - x(2x^2 - 9) + 7(2x^2 - 9)$

$= 8x^4 - 36x^2 - 2x^3 + 9x + 14x^2 - 63$

$= 8x^4 - 2x^3 - 22x^2 + 9x - 63$

TRY TUTORIAL 4.1: Question 4

WORKED EXAMPLE 5 Multiplying polynomials to find specific terms

Find the coefficient of x^2 in the expansion of $(6x^2 + 3x - 2)(5x^2 - 9x + 4)$.

SOLUTION

$(6x^2 + 3x - 2)(5x^2 - 9x + 4)$ ⟵ Multiply the terms whose products give the terms in x^2.

Term in x^2 is $6x^2(4) + 3x(-9x) - 2(5x^2) = 24x^2 - 27x^2 - 10x^2$

$= -13x^2$

$\therefore$ Coefficient of x^2 is **−13**

TRY TUTORIAL 4.1: Questions 5–7

WORKED EXAMPLE 6 Finding unknown coefficients in identities

Given that $3x^2 - 11x - 33 = a(x-4)(x+2) + b(x+2) + c$ for all values of x, find the values of a, b and c.

SOLUTION

$3x^2 - 11x - 33 = a(x-4)(x+2) + b(x+2) + c$

Method 1: Substitute suitable values of x

Let $x = -2$: $3(-2)^2 - 11(-2) - 33 = a(0) + b(0) + c$ ⟵ Letting x be -2 leaves us with 1 unknown, c.

$c = 1$

Let $x = 4$: $3(4)^2 - 11(4) - 33 = a(0) + b(4+2) + c$ ⟵ Letting x be 4 leaves us with 1 unknown, b.

$-29 = 6b + 1$

$6b = -30$

$b = -5$

Let $x = 0$: $3(0) - 11(0) - 33 = a(-4)(2) + b(2) + c$ ⟵ Letting x be 0 leaves us with 1 unknown, a.

$-33 = -8a + (-5)(2) + 1$

$-33 = -8a - 9$

$8a = 24$

$a = 3$

$\therefore$ **$a = 3$, $b = -5$, $c = 1$**

Method 2: Equate coefficients of like powers of x

Coefficients of x^2: $3 = a$

$a = 3$

Coefficients of x: $-11 = a(-4+2) + b$

$-11 = 3(-2) + b$

$-11 = -6 + b$

$b = -5$

Constant terms: $-33 = a(-8) + b(2) + c$

$-33 = 3(-8) - 5(2) + c$

$-33 = -24 - 10 + c$

$c = 1$

$\therefore$ **$a = 3$, $b = -5$, $c = 1$**

TRY TUTORIAL 4.1: Question 8

WORKED EXAMPLE 7 **Finding unknown coefficients in identities**

Given that $4x^3 - 49x + 60 = (2x - 5)(x + 4)(ax + b)$ for all values of x, find the values of a and b.

SOLUTION

$4x^3 - 49x + 60 = (2x - 5)(x + 4)(ax + b)$

Equating coefficients of x^3,

$4 = 2a$

$a = 2$

Let $x = 0$: $0 - 0 + 60 = (-5)(4)(b)$ ⟵ We can use a combination of Methods 1 and 2 to find the values of the unknown coefficients.

$60 = -20b$

$b = -3$

$\therefore$ **$a = 2$, $b = -3$**

TRY TUTORIAL 4.1: Question 9

WORKED EXAMPLE 8 Dividing polynomials

Find the remainder when

(a) $6x^2 + 7x - 8$ is divided by $x - 2$,

(b) $4x^3 - x + 3$ is divided by $2x + 1$.

SOLUTION

(a)

$$\begin{array}{rr|l}
 & 6x + 19 & \leftarrow \text{quotient} \\
\text{divisor} \rightarrow x - 2) & 6x^2 + 7x - 8 & \leftarrow \text{dividend} \\
 & -(6x^2 - 12x) & \\
 & 19x - 8 & \\
 & -(19x - 38) & \\
 & 30 & \leftarrow \text{remainder}
\end{array}$$

∴ The remainder is **30**. ⟵ $6x^2 + 7x - 8 = (x - 2)(6x + 19) + 30$

(b)

$$\begin{array}{rr|l}
 & 2x^2 - x & \leftarrow \text{quotient} \\
\text{divisor} \rightarrow 2x + 1) & 4x^3 + 0x^2 - x + 3 & \leftarrow \text{dividend} \\
 & -(4x^3 + 2x^2) & \\
 & -2x^2 - x & \\
 & -(-2x^2 - x) & \\
 & 3 & \leftarrow \text{remainder}
\end{array}$$

∴ The remainder is **3**. ⟵ $4x^3 - x + 3 = (2x + 1)(2x^2 - x) + 3$

TRY TUTORIAL 4.1: Questions 10, 11

WORKED EXAMPLE 9 **Applying the Division Algorithm for Polynomials**

It is given that $8x^4 + 10x^3 - 13x^2 + 2 = (4x^2 - x + 1)Q(x) + R(x)$ for all real values of x. Find $Q(x)$ and $R(x)$.

SOLUTION

$$
\begin{array}{r|l}
 & 2x^2 + 3x - 3 \\
4x^2 - x + 1 & 8x^4 + 10x^3 - 13x^2 + 0x + 2 \\
 & -(8x^4 - 2x^3 + 2x^2) \\
 & \quad 12x^3 - 15x^2 + 0x \\
 & \quad -(12x^3 - 3x^2 + 3x) \\
 & \qquad -12x^2 - 3x + 2 \\
 & \qquad -(-12x^2 + 3x - 3) \\
 & \qquad\qquad -6x + 5
\end{array}
$$

← Stop dividing when the degree of the remainder is less than that of the divisor.

$8x^4 + 10x^3 - 13x^2 + 2 = (4x^2 - x + 1)(2x^2 + 3x - 3) + 5 - 6x$

$\therefore$ **$Q(x) = 2x^2 + 3x - 3$ and $R(x) = 5 - 6x$**

TRY TUTORIAL 4.1: Questions 12–16

NAME: ____________________ CLASS: ____________ DATE: __________

TUTORIAL 4.1

1. Determine whether each of the following is a polynomial. Explain your answer briefly.

   (a) $6-\frac{1}{2}x$ (b) $(3x+4)^2$

   (c) $x^2+16\sqrt[3]{x}$ (d) πx^5

2. It is given that $P(x) = 3x^3 + 10x - 2$ and $Q(x) = 2x^2 - 6x + 9$.

   (i) Find an expression for each of the following and state its degree.

   (a) $P(x) + Q(x)$ (b) $2P(x) - 5Q(x)$

   (ii) Show that $2P(-2) - 5Q(-2) = -237$.

3. Suggest a pair of polynomials $P(x)$ and $Q(x)$ for which the degree of $P(x) + Q(x)$ is less than the degree of $P(x) - Q(x)$.

4. Expand each of the following.

   (a) $(3x + 5)(x^2 - 8x - 7)$ (b) $(6x^2 + x + 2)(10 - x^2)$

5. Find the coefficient of x^3 in the expansion of $(5x^2 - x + 1)(2x^2 + 3x - 9)$.

6. Find the value of the constant k for which the coefficient of x^2 in the expansion of $(3x^2 + 4x - k)(7x^2 - 8x + 6)$ is 0.

7. Suggest a quadratic polynomial $f(x)$ for which the product of $2x^2 - 6x - 1$ and $f(x)$ does not contain a term in x^3.

8. Find the values of the unknowns in each of the following.

   (a) $A(x + 5)(x - 2) + B(x - 2) = 4x^2 + 8x - 32$

   (b) $A(3x + 1) + B(1 - 3x) = 10$

   (c) $10x^2 - 33x - 20 = A(x + 1)(4 - x) + B(4 - x) + C$

   (d) $A - x^2 = 12 + B(x - 3) + C(x - 3)^2$

9. Given that $5x^3 - 27x^2 - 110x + 48 = (5x - 2)(x + 3)(ax + b)$ for all values of x, find the values of a and b.

10. Find the remainder when $12x^3 + 25x^2 + 3x - 12$ is divided by

    (a) $x - 1$, (b) $4x + 7$.

11. By using long division, divide $x^4 - x^3 - 9x^2 + 23x - 10$ by $x^2 + 3x - 2$.

12. **(i)** Divide $12x^3 + 5x^2 + 6x - 8$ by $4x - 1$ and state the remainder.

(ii) Hence, express $12x^3 + 5x^2 + 6x - 8$ in terms of $4x - 1$.

13. It is given that $6x^3 - 10x^2 + 7x - 4 = (2x^2 + 1)\mathrm{Q}(x) + \mathrm{R}(x)$ for all real values of x. Find $\mathrm{Q}(x)$ and $\mathrm{R}(x)$.

14. It is given that $-x^4 + 3x^3 + 6x^2 - 4 = (x^2 - 9)\mathrm{Q}(x) + Ax + B$, where $\mathrm{Q}(x)$ is a polynomial.

(i) State the degree of $\mathrm{Q}(x)$.

(ii) Find the values of the constants A and B.

15. Find the quotient and the remainder when $5x^4 + 10x^3y - 6xy^2 - 2y^3$ is divided by $x + 2y$.

16. State whether each of the following is true, false, or if there is insufficient information to conclude. $\mathrm{f}(x) \neq \mathrm{g}(x)$.

(a) If $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are polynomials, then $\mathrm{f}(x) \times \mathrm{g}(x)$ is a polynomial.

(b) If $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are polynomials, then $\mathrm{f}(x) \div \mathrm{g}(x)$ is a polynomial.

(c) If $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are non-polynomials, then $\mathrm{f}(x) + \mathrm{g}(x)$ is a non-polynomial.

(d) If $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are non-polynomials, then $\mathrm{f}(x) \times \mathrm{g}(x)$ is a non-polynomial.

4.2 Remainder and Factor Theorems

Objectives Checklist

- Use the Remainder Theorem to find the remainder when a polynomial is divided by a linear factor
- Use the Factor Theorem to factorise cubic expressions

Notes and Worked Examples

The **Remainder Theorem** states that:

> If a polynomial $f(x)$ is divided by a linear divisor $ax + b$, the remainder is $f\left(-\frac{b}{a}\right)$.

In particular, if a polynomial $f(x)$ is divided by a linear divisor $x + b$, the remainder is $f(-b)$.

The **Factor Theorem** states that:

> $ax + b$ is a factor of a polynomial $f(x) \Leftrightarrow f\left(-\frac{b}{a}\right) = 0$.

If $ax + b$ is a factor of $f(x)$, then $f(x)$ is exactly divisible by $ax + b$.

WORKED EXAMPLE 1 **Finding the remainder using the Remainder Theorem**

Find the remainder when $8x^3 - x^2 - 5x + 6$ is divided by

(a) $x - 2$,

(b) $2x + 3$.

SOLUTION

Let $f(x) = 8x^3 - x^2 - 5x + 6$.

(a) $f(2) = 8(2)^3 - 2^2 - 5(2) + 6$
$= 56$

$\therefore$ The remainder is **56**.

(b) $f\left(-\frac{3}{2}\right) = 8\left(-\frac{3}{2}\right)^3 - \left(-\frac{3}{2}\right)^2 - 5\left(-\frac{3}{2}\right) + 6$
$= -\frac{63}{4}$

$\therefore$ The remainder is $-\mathbf{\frac{63}{4}}$.

TRY TUTORIAL 4.2: Questions 1, 2

WORKED EXAMPLE 2 **Finding the value of an unknown using the Remainder Theorem**

The expression $8x^5 - px^3 + x^2 - 3$, where p is a constant, leaves a remainder of -4 when divided by $2x - 1$. Find the value of p.

SOLUTION

Let $f(x) = 8x^5 - px^3 + x^2 - 3$.

By the Remainder Theorem,

$$f\left(\frac{1}{2}\right) = -4$$

$$8\left(\frac{1}{2}\right)^5 - p\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 - 3 = -4$$

$$\frac{1}{4} - \frac{1}{8}p + \frac{1}{4} - 3 = -4$$

$$\frac{1}{8}p = \frac{3}{2}$$

$$p = \mathbf{12}$$

TRY TUTORIAL 4.2: Questions 3–6

WORKED EXAMPLE 3 Finding the values of unknowns using the Remainder Theorem

The expression $-5x^3 + ax^2 + bx + c$ leaves the same remainder when divided by $x + 2$ or by $x - 1$.

(i) Show that $a - b = -15$.

(ii) Give an example of the values of a and b which satisfy the conditions found in part **(i)**.

SOLUTION

(i) Let $f(x) = -5x^3 + ax^2 + bx + c$.

Since $f(-2) = f(1)$,

$$-5(-2)^3 + a(-2)^2 + b(-2) + c = -5(1)^2 + a(1)^2 + b(1) + c$$

$$40 + 4a - 2b + c = -5 + a + b + c$$

$$3a - 3b = -45$$

$$a - b = -15 \text{ (shown)}$$

(ii) A possible pair of values is $\boldsymbol{a = 5}$ and $\boldsymbol{b = 20}$.

TRY TUTORIAL 4.2: Questions 7, 8

WORKED EXAMPLE 4 Determining whether $ax + b$ is a factor of a polynomial

(a) Determine whether $x - 3$ is a factor of $x^3 + x^2 - 11x - 3$.

(b) Show that $4x + 1$ is not a factor of $4x^3 + 6x - 1$.

SOLUTION

(a) Let $f(x) = x^3 + x^2 - 11x - 3$.

$$f(3) = 3^3 + 3^2 - 11(3) - 3$$
$$= 0$$

$\therefore$ By the Factor Theorem, $x - 3$ is a factor of $x^3 + x^2 - 11x - 3$.

(b) Let $g(x) = 4x^3 + 6x - 1$.

$$g\left(-\frac{1}{4}\right) = 4\left(-\frac{1}{4}\right)^3 + 6\left(-\frac{1}{4}\right) - 1$$
$$= -\frac{41}{16}$$

$\therefore$ Since $g\left(-\frac{1}{4}\right) \neq 0$, $4x + 1$ is not a factor of $4x^3 + 6x - 1$. (shown)

TRY TUTORIAL 4.2: Questions 9–11

WORKED EXAMPLE 5 **Finding the value of an unknown using the Factor Theorem**

The expression $px^3 + 10x^2 + 2x + 40$ has a factor $x + 4$. Find the value of p.

SOLUTION

Let $f(x) = px^3 + 10x^2 + 2x + 40$.

By the Factor Theorem,

$$f(-4) = 0$$
$$p(-4)^3 + 10(-4)^2 + 2(-4) + 40 = 0$$
$$-64p + 160 - 8 + 40 = 0$$
$$64p = 192$$
$$p = 3$$

TRY TUTORIAL 4.2: Questions 12, 13

WORKED EXAMPLE 6 **Finding the values of unknowns using the Factor Theorem**

The expression $ax^3 - 17x^2 + bx + 50$, where a and b are constants, is exactly divisible by $x^2 - 3x - 10$. Find the value of a and of b.

SOLUTION

Let $f(x) = ax^3 - 17x^2 + bx + 50$.

Since $x^2 - 3x - 10 = (x - 5)(x + 2)$, $f(x)$ is divisible by $x - 5$ and $x + 2$.

By the Factor Theorem, $f(5) = 0$ and $f(-2) = 0$.

$$f(5) = a(5)^3 - 17(5)^2 + b(5) + 50 = 0$$
$$125a - 425 + 5b + 50 = 0$$
$$125a + 5b = 375$$
$$25a + b = 75 \quad \text{— (1)}$$

$$f(-2) = a(-2)^3 - 17(-2)^2 + b(-2) + 50 = 0$$
$$-8a - 68 - 2b + 50 = 0$$
$$8a + 2b = -18$$
$$4a + b = -9 \quad \text{— (2)}$$

(1) – (2):
$$21a = 84$$
$$a = 4$$

Substitute $a = 4$ into (2):
$$4(4) + b = -9$$
$$16 + b = -9$$
$$b = -25$$

$\therefore$ **$a = 4$, $b = -25$**

TRY TUTORIAL 4.2: Questions 14–16

NAME: ______________________ CLASS: __________ DATE: __________

TUTORIAL 4.2

1. Find the remainder when $9x^3 - 2x^2 - 2x + 4$ is divided by

(a) $x + 3$, **(b)** $3x - 1$.

2. Find the remainder when $\frac{1}{2}(5x^2 + 1)(x^2 - 6)$ is divided by

(a) $1 - x$, **(b)** x.

3. When $4x^3 + a(x^2 + x) - 10$ is divided by $2x - 1$, the remainder is $-\frac{29}{4}$. Find the value of a.

4. The remainder is 1 when $2x^7 - (a + 2)x^2 + x + 4$ is divided by $x + 1$. Find the value of a.

5. The function f is defined by $f(x) = 16x^4 + 8ax^3 - 6x^2 - 28x + a$, where a is a constant. When $f(x)$ is divided by $2x + a$, the remainder is 36. Find the possible values of a.

6. The remainder when $\frac{1}{4}ax^3 - 6$ is divided by $2x + 3$ is equal to the remainder when $3x^3 + ax^2 - 7$ is divided by $2x - 1$. Find the value of a.

7. The expression $ax^6 + 4x^3 - bx + 2c$ leaves the same remainder when divided by $x - 2$ or by $x + 1$.

(i) Show that $21a - b = -12$.

(ii) Give an example of the values of a and b which satisfy the conditions found in part **(i)**.

8. When $px^3 + qx^2 + px + q$ is divided by $2x - 1$, the remainder is twice of that when it is divided by $2x + 1$. Find the value of $\frac{4p}{7q}$.

9. Determine whether $x + 3$ is a factor of $x^3 - 4x^2 - 11x + 28$.

10. Show that $5x - 2$ is a factor of $5x^3 - 7x^2 - 8x + 4$.

11. The function f is defined by $f(x) = x^4 - 9x^2 + kx - 7$, where k is a constant.

(i) Given that $x - 3$ is a factor of $f(x)$, find the value of k.

(ii) Using the value of k found in part **(i)**, show that $x + 3$ is not a factor of $f(x)$.

12. The expression $x^3 + px^2 + 5x - 50$ has a factor $x + 5$. Find the value of p.

13. The expression $\frac{1}{3}x^3 - px^2 + 4$ has a factor $2x + 3$.

(i) Find the value of p.

(ii) Hence, find the remainder when $\frac{1}{3}x^3 - px^2 + 4$ is divided by x.

14. The expression $ax^3 + 10x^2 + bx - 24$, where a and b are constants, is exactly divisible by $x^2 + 7x - 8$. Find the value of a and of b.

15. The expressions $14x^3 + px^2 - 11x + q$ and $2x^2 + 3x - 5$ have common factors. Find the values of p and q.

16. Two factors of a polynomial of degree 3 are $4x + 1$ and $x - 5$. When the polynomial is divided by $x - 3$, the remainder is -52. Find a possible expression for this polynomial.

4.3 Cubic expressions, equations and identities

Objectives Checklist

- Apply the cubic identities $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
- State the types of roots of cubic equations
- Solve cubic equations and problems involving cubic equations

Notes and Worked Examples

A cubic expression in x is a polynomial of degree 3, of the form $ax^3 + bx^2 + cx + d$, where $a \neq 0$.

We use the Factor Theorem to factorise a cubic expression:

Step 1: Use the Factor Theorem to find a linear factor of $f(x)$, e.g. $x - p$.

Step 2: Write the cubic expression in the form $(x - p)(ax^2 + bx + c)$.

Step 3: Substitute suitable values of x or equate coefficients of like powers of x to find the values of a, b and c.

Step 4: Factorise $ax^2 + bx + c$ if possible.

To solve a cubic equation:

Step 5: Equate each factor of the cubic expression to 0.

The **sum and difference of cubes** are:

- $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
- $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

WORKED EXAMPLE 1 Factorising a cubic expression involving $a^3 \pm b^3$

Factorise each of the following expressions.

(a) $x^3 + 8$

(b) $x^3 - 8$

(c) $27x^3 + 64y^6$

(d) $2(x + 5)^3 - 250x^3$

SOLUTION

(a) $x^3 + 8 = x^3 + 2^3$

$= \mathbf{(x + 2)(x^2 - 2x + 4)}$

(b) $x^3 - 8 = x^3 - 2^3$

$= \mathbf{(x - 2)(x^2 + 2x + 4)}$

(c) $27x^3 + 64y^6 = (3x)^3 + (4y^2)^3$

$= \mathbf{(3x + 4y^2)(9x^2 - 12xy^2 + 16y^4)}$

(d) $2(x + 5)^3 - 250x^3 = 2[(x + 5)^3 - 125x^3]$

$= 2[(x + 5)^3 - (5x)^3]$

$= 2(x + 5 - 5x)[(x + 5)^2 + (x + 5)(5x) + (5x)^2]$

$= 2(5 - 4x)(x^2 + 10x + 25 + 5x^2 + 25x + 25x^2)$

$= \mathbf{2(5 - 4x)(31x^2 + 35x + 25)}$

TRY TUTORIAL 4.3: Questions 1–3

WORKED EXAMPLE 2 **Factorising a cubic expression**

Factorise $4x^3 + 13x^2 - 37x - 10$ completely.

SOLUTION

Let $f(x) = 4x^3 + 13x^2 - 37x - 10$.

$f(2) = 4(2)^3 + 13(2)^2 - 37(2) - 10$ ← Find a value of p such that $f(p) = 0$.

$= 0$

$\therefore$ By the Factor Theorem, $x - 2$ is a factor of $f(x)$.

$4x^3 + 13x^2 - 37x - 10 = (x - 2)(4x^2 + bx + 5)$

Equating coefficients of x^2, ← Equating coefficients of x will also yield $b = 21$.

$13 = b - 8$

$b = 21$

$4x^3 + 13x^2 - 37x - 10 = (x - 2)(4x^2 + 21x + 5)$

$= \mathbf{(4x + 1)(x + 5)(x - 2)}$

TRY TUTORIAL 4.3: Questions 4–7

WORKED EXAMPLE 3 **Solving a cubic equation**

Solve each of the following equations, leaving your answer in surd form where necessary.

(a) $x^3 - 8x^2 + 9x + 18 = 0$ **(b)** $3x^3 + 4x^2 = 16x + 8$

SOLUTION

(a) Let $f(x) = x^3 - 8x^2 + 9x + 18$.

$f(-1) = (-1)^3 - 8(-1)^2 + 9(-1) + 18$

$= 0$

$\therefore$ By the Factor Theorem, $x + 1$ is a factor of $f(x)$.

$x^3 - 8x^2 + 9x + 18 = (x + 1)(x^2 + bx + 18)$

Equating coefficients of x^2, ← Equating coefficients of x will also yield $b = -9$.

$-8 = b + 1$

$b = -9$

$x^3 - 8x^2 + 9x + 18 = (x + 1)(x^2 - 9x + 18)$

$= (x + 1)(x - 3)(x - 6)$

When $f(x) = 0$,

$(x + 1)(x - 3)(x - 6) = 0$

$x = -1$ or $x = 3$ or $x = 6$

$\therefore x = \mathbf{-1}$ or $x = \mathbf{3}$ or $x = \mathbf{6}$

(b) $3x^3 + 4x^2 = 16x + 8$

$3x^3 + 4x^2 - 16x - 8 = 0$

Let $g(x) = 3x^3 + 4x^2 - 16x - 8$.

$g(2) = 3(2)^3 + 4(2)^2 - 16(2) - 8$

$= 0$

$\therefore$ By the Factor Theorem, $x - 2$ is a factor of $g(x)$.

$3x^3 + 4x^2 - 16x - 8 = (x - 2)(3x^2 + bx + 4)$

Equating coefficients of x, ⟵ Equating coefficients of x^2 will also yield $b = 10$.

$-16 = 4 - 2b$

$2b = 20$

$b = 10$

$3x^3 + 4x^2 - 16x - 8 = (x - 2)(3x^2 + 10x + 4)$

When $g(x) = 0$,

$(x - 2)(3x^2 + 10x + 4) = 0$

$x = 2$ or $x = \dfrac{-10 \pm \sqrt{10^2 - 4(3)(4)}}{2(3)}$

$= \dfrac{-10 \pm \sqrt{52}}{6}$

$= \dfrac{-10 \pm 2\sqrt{13}}{6}$

$= \dfrac{-5 \pm \sqrt{13}}{3}$

$\therefore x = 2$ or $x = \dfrac{\sqrt{13} - 5}{3}$ or $x = \dfrac{-\sqrt{13} - 5}{3}$

TRY TUTORIAL 4.3: Questions 8, 9

WORKED EXAMPLE 4 **Solving a cubic equation**

The equation of a polynomial is given by $p(x) = 25x^3 + 19x - 4$.

(i) Find the remainder when $p(x)$ is divided by $x + 3$.

(ii) Show that $5x - 1$ is a factor of $p(x)$.

(iii) Show that the equation $p(x) = 0$ has only one real root.

(iv) Use your answers to parts **(ii)** and **(iii)** to solve the equation $5^{3y+2} + 19(5^y) = 4$.

SOLUTION

(i) $p(x) = 25x^3 + 19x - 4$

$p(-3) = 25(-3)^3 + 19(-3) - 4$

$= -736$

$\therefore$ The remainder is **−736**.

(ii) $p\left(\frac{1}{5}\right) = 25\left(\frac{1}{5}\right)^3 + 19\left(\frac{1}{5}\right) - 4$

$= 0$

$\therefore$ By the Factor Theorem, $5x - 1$ is a factor of $25x^3 + 19x - 4$. (shown)

(iii) $25x^3 + 19x - 4 = (5x - 1)(5x^2 + bx + 4)$

Equating coefficients of x^2, ⟵ Equating coefficients of x will also yield $b = 1$.

$0 = 5b - 5$

$5b = 5$

$b = 1$

$25x^3 + 19x - 4 = (5x - 1)(5x^2 + x + 4)$

When $p(x) = 0$,

$(5x - 1)(5x^2 + x + 4) = 0$

$x = \frac{1}{5}$ or $5x^2 + x + 4 = 0$

Discriminant $= 1^2 - 4(5)(4)$

$= -79 < 0$

$\therefore$ $p(x) = 0$ has only one real root. (shown)

(iv) $5^{3y+2} + 19(5^y) = 4$

$25(5^y)^3 + 19(5^y) - 4 = 0$ ← Manipulate the equation such that it is in the same form as $p(x) = 0$.

From **(iii)**,

$5^y = \frac{1}{5}$

$= 5^{-1}$

$y = \mathbf{-1}$

TRY TUTORIAL 4.3: Questions 10–14

NAME: ____________________ CLASS: __________ DATE: __________

TUTORIAL **4.3**

1. Factorise each of the following expressions.

(a) $x^3 + 27$

(b) $x^3 - 1000$

(c) $64x^3 + y^6$

(d) $x^6 - 125y^9$

(e) $(2x + y)^3 + (2x - y)^3$

(f) $2(x - y)^3 - 16y^3$

2. **(i)** Show that $a - b$ is a factor of $a^3 - b^3$.

(ii) Hence, factorise $a^3 - b^3$.

(iii) Using your answer in part **(ii)**, factorise $8x^6 - 343y^{12}$.

3. **(i)** Using the substitution $u = 27x^3$, or otherwise, express $729x^6 - 64$ as the product of two factors.

(ii) Hence, express $729x^6 - 64$ as the product of four factors with integer coefficients.

4. Factorise each of the following expressions completely.

(a) $x^3 - 6x^2 + 11x - 6$

(b) $x^3 + 3x^2 - 24x - 80$

(c) $5x^3 + 16x^2 - x - 12$

(d) $-14x^3 - 9x^2 + 32x - 12$

5. The polynomial $f(x)$ is given by $f(x) = 5x^3 + 14x^2 - 23x + 4$.

(i) Show that $5x - 1$ is a factor of $f(x)$.

(ii) Express $f(x)$ as a product of three linear factors.

6. **(i)** Show that $x + 3$ is a factor of $x^3 - 3x^2 - 10x + 24$.

(ii) Factorise $x^3 - 3x^2 - 10x + 24$ completely.

7. The polynomial $f(x)$ is given by $f(x) = 6x^3 - 11x^2 - 50x - 8$.

(i) Find the remainder when $f(x)$ is divided by x.

(ii) Show that $(x - 4)(x + 2)$ is a factor of $f(x)$.

(iii) Express $f(x)$ as a product of three linear factors.

8. Solve each of the following equations, leaving your answer in surd form where necessary.

(a) $x^3 - 7x^2 - 10x + 16 = 0$

(b) $9x^3 - 30x^2 - 23x - 4 = 0$

(c) $8x^3 - 12x^2 + 6x - 1 = 0$

(d) $4x^3 - x^2 - 28x - 15 = 0$

9. **(i)** Find the value of a and of b for which $6x^2 - 5x + 1$ is a factor of $6x^4 + x^3 + a(x^2 - x) + b$.

(ii) Using the values of a and b found in part **(i)**, solve the equation

$$6x^4 + x^3 + a(x^2 - x) + b = 0.$$

10. The equation of a polynomial is given by $p(x) = 4x^3 + 7x^2 + 18x - 5$.

(i) Find the remainder when $p(x)$ is divided by $x + 3$.

(ii) Show that $4x - 1$ is a factor of $p(x)$.

(iii) Show that the equation $p(x) = 0$ has only one real root.

(iv) Use your answers to parts **(ii)** and **(iii)** to solve the equation

$$2^{3y+2} + 7(2^{2y}) + 9(2^{y+1}) = 5.$$

11. Factorise $125x^3 - 27$ and hence prove that $x^3 = \frac{27}{125}$ has only one real solution.

12. The function $f(x) = x^3 + ax^2 + x + b$, where a and b are constants, is exactly divisible by $x + 5$. Given that $f(x)$ leaves a remainder of 26 when divided by $x + 4$,

(i) find the value of a and of b,

(ii) determine, showing all necessary working, the number of real roots of the equation $f(x) = 0$.

13. **(i)** Solve the equation $x^3 + 3x^2 - 6x - 8 = 0$.

(ii) On the same diagram, sketch the curve $y = x^3 + 3x^2 - 4x$ and the line $y = 2x + 8$. Indicate clearly the coordinates of the points of intersection.

(iii) The equation $x^3 + 3x^2 - 4x = ax + b$ has exactly one real root. Suggest a pair of values of a and b.

14. The term containing the highest power of x in the polynomial $f(x)$ is $3x^4$. Two of the roots of the equation $f(x) = 0$ are -1 and 4. Given that $x^2 - 3x + 6$ is a quadratic factor of $f(x)$, find

(i) an expression for $f(x)$ in descending powers of x,

(ii) the number of real roots of the equation $f(x) = 0$, justifying your answer.

The straight line L meets the curve $y = f(x)$ at one point only.

(iii) Given that L is not a tangent to the curve, what can be deduced about L?

4.4 Partial fractions

Objectives Checklist

- Express algebraic fractions in partial fractions

Notes and Worked Examples

An algebraic fraction is of the form $\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}$, where $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ are polynomials and $\mathrm{Q}(x) \neq 0$.

If the degree of $\mathrm{P}(x)$ is less than the degree of $\mathrm{Q}(x)$, then $\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ is a **proper fraction**.

If the degree of $\mathrm{P}(x)$ is greater than or equal to the degree of $\mathrm{Q}(x)$, then $\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ is an **improper fraction**.

To express an algebraic fraction in partial fractions:

Step 1: If the algebraic fraction is an improper fraction, use long division to express it as the sum of a polynomial and a proper fraction $\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}$, then consider the proper fraction in the next steps.

Step 2: Factorise the denominator $\mathrm{Q}(x)$ completely.

Step 3: Express $\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ in partial fractions according to the table below.

Step 4: Solve for the unknowns by substituting suitable values of x and/or comparing coefficients of like terms.

Case	Denominator contains	Corresponding partial fraction	Example
1	linear factor $ax + b$	$\frac{A}{ax+b}$	$\frac{x-10}{(x+4)(x-3)} = \frac{2}{x+4} - \frac{1}{x-3}$
2	repeated linear factor $(ax + b)^2$	$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$	$\frac{2x+1}{(2x-5)^2} = \frac{1}{2x-5} + \frac{6}{(2x-5)^2}$
3	quadratic factor $x^2 + c^2$ (which cannot be factorised)	$\frac{Ax+B}{x^2+c^2}$	$\frac{14-11x}{(3x-2)(x^2+4)} = \frac{3}{2(3x-2)} - \frac{x+8}{2(x^2+4)}$

WORKED EXAMPLE 1 **Proper fraction with distinct linear factors in the denominator**

Express $\dfrac{x-10}{x^2+x-12}$ in partial fractions.

SOLUTION

$\dfrac{x-10}{x^2+x-12} = \dfrac{x-10}{(x+4)(x-3)}$ ⟵ Factorise the denominator.

Let $\dfrac{x-10}{(x+4)(x-3)} = \dfrac{A}{x+4}+\dfrac{B}{x-3}$.

$x - 10 = A(x-3) + B(x+4)$ ⟵ Multiply throughout by $(x+4)(x-3)$.

Let $x = -4$: $-4 - 10 = A(-4-3) + B(0)$ ⟵ Letting x be -4 leaves us with 1 unknown, A.

$-14 = -7A$

$A = 2$

Let $x = 3$: $3 - 10 = A(0) + B(3+4)$ ⟵ Alternatively, equate the coefficients of x: $1 = A + B$

$-7 = 7B$

$B = -1$

$\therefore \dfrac{x-10}{x^2+x-12} = \dfrac{2}{x+4} - \dfrac{1}{x-3}$

TRY TUTORIAL 4.4: Question 1

WORKED EXAMPLE 2 **Proper fraction with repeated linear factors in the denominator**

Express $\frac{2x+1}{(2x-5)^2}$ in partial fractions.

SOLUTION

$$\frac{2x+1}{(2x-5)^2} = \frac{A}{2x-5} + \frac{B}{(2x-5)^2}$$

$2x + 1 = A(2x - 5) + B$ ← Multiply throughout by $(2x - 5)^2$.

Let $x = \frac{5}{2}$: $2\left(\frac{5}{2}\right)+1 = A(0) + B$ ← Letting x be $\frac{5}{2}$ leaves us with 1 unknown, B.

$6 = B$

$B = 6$

Let $x = 0$: $2(0) + 1 = A(-5) + B$ ← Alternatively, equate the coefficients of x: $2 = 2A$

$1 = -5A + 6$

$5A = 5$

$A = 1$

$$\therefore \frac{2x+1}{(2x-5)^2} = \frac{1}{2x-5} + \frac{6}{(2x-5)^2}$$

TRY TUTORIAL 4.4: Question 2

WORKED EXAMPLE 3 **Proper fraction with repeated linear factors in the denominator**

Express $\frac{2x^2+6x+7}{(x-2)(x+1)^2}$ in partial fractions.

SOLUTION

Let $\frac{2x^2+6x+7}{(x-2)(x+1)^2} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$.

$2x^2 + 6x + 7 = A(x+1)^2 + B(x-2)(x+1) + C(x-2)$ ← Multiply throughout by $(x-2)(x+1)^2$.

Let $x = 2$: $2(2)^2 + 6(2) + 7 = A(2+1)^2 + B(0) + C(0)$ ← Letting x be 2 leaves us with 1 unknown, A.

$27 = 9A$

$A = 3$

Let $x = -1$: $2(-1)^2 + 6(-1) + 7 = A(0) + B(0) + C(-1-2)$ ← Letting x be -1 leaves us with 1 unknown, C.

$3 = -3C$

$C = -1$

Let $x = 0$: $0 + 0 + 7 = A(1)^2 + B(-2)(1) + C(-2)$ ← Letting x be 0 leaves us with 1 unknown, B.

$7 = A - 2B - 2C$

$7 = 3 - 2B - 2(-1)$

$2B = -2$

$B = -1$

$\therefore \frac{2x^2+6x+7}{(x-2)(x+1)^2} = \frac{3}{x-2} - \frac{1}{x+1} - \frac{1}{(x+1)^2}$

TRY TUTORIAL 4.4 Questions 3, 5

WORKED EXAMPLE 4 Proper fraction with quadratic factor (which cannot be factorised) in the denominator

Express $\dfrac{14-11x}{(3x-2)(x^2+4)}$ in partial fractions.

SOLUTION

Let $\dfrac{14-11x}{(3x-2)(x^2+4)} = \dfrac{A}{3x-2}+\dfrac{Bx+C}{x^2+4}$. ⟵ $x^2+4 \neq (x+2)^2$

$14 - 11x = A(x^2 + 4) + (Bx + C)(3x - 2)$ ⟵ Multiply throughout by $(3x-2)(x^2+4)$.

Let $x = \dfrac{2}{3}$: $14-11\left(\dfrac{2}{3}\right) = A\left[\left(\dfrac{2}{3}\right)^2+4\right]+0$ ⟵ Letting x be $\dfrac{2}{3}$ leaves us with 1 unknown, A.

$\dfrac{20}{3} = \dfrac{40}{9}A$

$A = \dfrac{3}{2}$

Let $x = 0$: $14 - 11(0) = \dfrac{3}{2}(0+4)+C(-2)$ ⟵ Letting x be 0 leaves us with 1 unknown, C.

$14 = 6 - 2C$

$2C = -8$

$C = -4$

Equating the coefficients of x^2,

$0 = A + 3B$

$0 = \dfrac{3}{2}+3B$

$3B = -\dfrac{3}{2}$

$B = -\dfrac{1}{2}$

$$\therefore \frac{14-11x}{(3x-2)(x^2+4)} = \frac{\frac{3}{2}}{3x-2}+\frac{-\frac{1}{2}x-4}{x^2+4}$$

$$= \frac{3}{2(3x-2)} - \frac{x+8}{2(x^2+4)}$$

TRY TUTORIAL 4.4: Question 4

WORKED EXAMPLE 5

Expressing an improper fraction as the sum of a polynomial and partial fractions

Express $\frac{5x^3+9x^2-40x-2}{(x+4)(x-2)}$ as the sum of a polynomial and 2 partial fractions.

SOLUTION

$\frac{5x^3+9x^2-40x-2}{(x+4)(x-2)} = \frac{5x^3+9x^2-40x-2}{x^2+2x-8}$ ⟵ Degree of numerator > Degree of denominator

$$\begin{array}{r|l} & 5x - 1 \\ \hline x^2+2x-8 & 5x^3 + 9x^2 - 40x - 2 \\ & -(5x^3 + 10x^2 - 40x) \\ \hline & -x^2 + 0x - 2 \\ & -(-x^2 - 2x + 8) \\ \hline & 2x - 10 \end{array}$$

$\therefore \frac{5x^3+9x^2-40x-2}{(x+4)(x-2)} = 5x-1+\frac{2x-10}{(x+4)(x-2)}$

Let $\frac{2x-10}{(x+4)(x-2)} = \frac{A}{x+4}+\frac{B}{x-2}$.

$2x - 10 = A(x-2) + B(x+4)$ ⟵ Multiply throughout by $(x+4)(x-2)$.

Let $x = -4$: $2(-4) - 10 = A(-4-2) + B(0)$ ⟵ Letting x be -4 leaves us with 1 unknown, A.

$-18 = -6A$

$A = 3$

Let $x = 2$: $2(2) - 10 = A(0) + B(2+4)$ ⟵ Alternatively, equate the coefficients of x: $2 = A + B$

$-6 = 6B$

$B = -1$

$\therefore \frac{2x-10}{(x+4)(x-2)} = \frac{3}{x+4}-\frac{1}{x-2}$

$\therefore \frac{5x^3+9x^2-40x-2}{(x+4)(x-2)} = 5x-1+\frac{3}{x+4}-\frac{1}{x-2}$

TRY TUTORIAL 4.4: Questions 6–10

NAME: ______________________ CLASS: __________ DATE: __________

TUTORIAL 4.4

1. Express each of the following in partial fractions.
   (a) $\frac{23x+4}{(3x-1)(x+2)}$ (b) $\frac{17}{(8x-3)(4x+7)}$
   (c) $\frac{5x-26}{x^2-4}$ (d) $\frac{1}{30x^2+6x}$

2. Express each of the following in partial fractions.
   (a) $\frac{12x+14}{(4x+7)^2}$ (b) $\frac{40x-23}{25x^2-30x+9}$

3. Express each of the following in partial fractions.
   (a) $\frac{x^2-24x-27}{x^2(x-9)}$ (b) $\frac{x^2-42x+159}{(x+3)(x-4)^2}$

4. Express each of the following in partial fractions.
   (a) $\frac{2x^2+x+2}{x(x^2+1)}$ (b) $\frac{2x-4}{(x^2+4)(x+2)}$
   (c) $\frac{14x^2-7x+19}{(3x-1)(x^2+9)}$ (d) $\frac{35-8x^2}{5x^3+25x}$

5. Express $\frac{(x+4)^2}{x^2(x-4)}$ as the sum of three partial fractions.

6. Express each of the following as the sum of a polynomial and a proper fraction.
   (a) $\frac{8x^2+18x-57}{(2x+7)(x-2)}$ (b) $\frac{9x^3+3x^2}{9x^2-6x+1}$

7. Express $\frac{2x^2-59}{x^2-x-20}$ as the sum of a polynomial and partial fractions.

8. (i) Using long division, divide $x^3 - 2x^2 - 12x - 15$ by $9 - x^2$.
   (ii) Hence, express $\frac{15+12x+2x^2-x^3}{x^2-9}$ as the sum of a polynomial and partial fractions.

9. (i) Show that $x - 4$ is a factor of $x^3 - 4x^2 + 16x - 64$.
   (ii) Express $\frac{3x^3-13x^2+32x-272}{x^3-4x^2+16x-64}$ as the sum of a polynomial and partial fractions.

10. Aaron wrote that $\frac{x^2-x+12}{(x+1)(x^2+5)}$ is equivalent to $\frac{7}{x+1}-\frac{4x-1}{x^2+5}$.

(i) Show how it is possible to verify that Aaron is incorrect without finding the partial fractions of $\frac{x^2-x+12}{(x+1)(x^2+5)}$.

(ii) Find the correct partial fractions of $\frac{x^2-x+12}{(x+1)(x^2+5)}$.

NAME: ______________________ CLASS: __________ DATE: ______

Quick Test 4

[Total marks: 25]

1. A polynomial f(x) is such that, when it is divided by $x + 1$ and $3x - 4$, the remainders are -5 and 2 respectively. Find the remainder when f(x) is divided by $3x^2 - x - 4$. [4]

2. The function f(x) is defined by $f(x) = x^3 + ax^2 + bx + 16$ for all real x. Given that $x + 4$ is a factor of f(x) and that when f(x) is divided by $x + 3$ the remainder is 1,

   (i) find the value of each of the constants a and b, [4]

   (ii) express f(x) as the product of linear factors. [3]

   (iii) Hence, find the number of real roots of the equation $x^6 + ax^4 + bx^2 + 16 = 0$. [2]

3. Given that $f(x) = 3x^3 - 17x^2 + 18x + 8$,

(i) find the remainder when $f(x)$ is divided by $x + 6$, [2]

(ii) show that $x - 4$ is a factor of $f(x)$ and hence solve the equation $f(x) = 0$. [4]

4. **(i)** By using long division, divide $2x^3 - x^2 + 8x - 4$ by $2x - 1$. [1]

(ii) Express $\dfrac{9x+4}{2x^3 - x^2 + 8x - 4}$ in partial fractions. [5]