CHAPTER 6 Exponential and Logarithmic Functions

O LEVEL ONLY

Chapter Summary

Indices — $y = a^x$ is equivalent to $x = \log_a y$ — **Logarithms**

Indices

- **Law 1:** $a^m \times a^n = a^{m+n}$ ↔ **Product Law:** $\log_a x + \log_a y = \log_a xy$
- **Law 2:** $\frac{a^m}{a^n} = a^{m-n}$ ↔ **Quotient Law:** $\log_a x - \log_a y = \log_a \frac{x}{y}$
- **Law 3:** $(a^m)^n = a^{mn}$ ↔ **Power Law:** $\log_a x^r = r \log_a x$
- **Law 4:** $a^n \times b^n = (ab)^n$
- **Law 5:** $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$

Change of Base Formula: $\log_a b = \frac{\log_c b}{\log_c a} = \frac{1}{\log_b a}$

definitions

- $a^0 = 1$
- $a^{-n} = \frac{1}{a^n}$
- $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

special properties

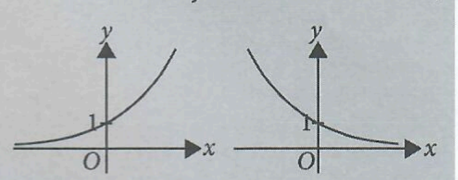
- $\log_a 1 = 0$
- $\log_a a = 1$
- $a^{\log_a b} = b$

exponential graphs — logarithmic graphs

$y = 10^x \Leftrightarrow x = \lg y$

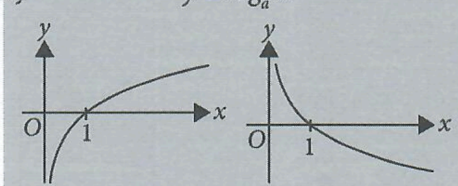
$y = e^x \Leftrightarrow x = \ln y$

$y = a^x$



$a > 1$ $\qquad$ $0 < a < 1$

$y = \log_a x$



$a > 1$ $\qquad$ $0 < a < 1$

Logarithmic and exponential equations

logarithmic equation

- The equation simplifies to $\log_a p = \log_a q \Leftrightarrow p = q$
- The equation simplifies to $\log_a x = b \Leftrightarrow x = a^b$

exponential equation

$a^x = b$

- b can be expressed as a^y: $a^x = a^y \Rightarrow x = y$
- b cannot be expressed as a^y: $x \lg a = \lg b \Rightarrow x = \dfrac{\lg b}{\lg a}$

$p(a^{2x}) + q(a^x) + r = 0$

To solve an exponential equation $p(a^{2x}) + q(a^x) + r = 0$:

Step 1: Let $u = a^x$ to get a quadratic equation in u.

Step 2: Factorise the expression on the LHS and solve for the value(s) of u.

Step 3: Replace u with a^x to solve for the value(s) of x.

6.1 Exponential expressions and equations

Objectives Checklist

- Simplify expressions involving exponents
- Solve equations involving exponents

Notes and Worked Examples

Recall that a^n ← index.
↑
base

If $a > 0$, $b > 0$, and m and n are real numbers, then the 5 **Laws of Indices** state that:

Law 1: $a^m \times a^n = a^{m+n}$

Law 2: $\dfrac{a^m}{a^n} = a^{m-n}$

Law 3: $(a^m)^n = a^{mn}$

Law 4: $a^n \times b^n = (ab)^n$

Law 5: $\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$

If $a > 0$ and $n > 0$, then the 3 **Definitions of Indices** state that:

Definition 1: $a^0 = 1$

Definition 2: $a^{-n} = \dfrac{1}{a^n}$

Definition 3: $a^{\frac{1}{n}} = \sqrt[n]{a}$

Extending Definition 3, we have:

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

WORKED EXAMPLE 1 Simplifying exponential expressions

Without using a calculator, simplify each of the following.

(a) $16^{\frac{1}{3}} \times 16^{\frac{1}{4}} \div 16^{\frac{7}{12}}$

(b) $(25^3 \div 5^3) \times 2^3$

(c) $(\sqrt[3]{3})^6 \times (\sqrt{3})^{-1} \div (\sqrt[4]{3})^{10}$

(d) $\left[(\sqrt[3]{49})^4 \div \left(\frac{1}{343}\right)^{-\frac{2}{3}}\right] \div 7^{-\frac{1}{3}}$

SOLUTION

(a) $16^{\frac{1}{3}} \times 16^{\frac{1}{4}} \div 16^{\frac{7}{12}} = 16^{\frac{1}{3}+\frac{1}{4}-\frac{7}{12}}$ ⟵ $a^m \times a^n = a^{m+n}$ and $a^m \div a^n = a^{m-n}$

$= 16^0$

$= \mathbf{1}$

(b) $(25^3 \div 5^3) \times 2^3 = (25 \div 5)^3 \times 2^3$ ⟵ $a^n \div b^n = (a \div b)^n$

$= 5^3 \times 2^3$

$= (5 \times 2)^3$ ⟵ $a^n \times b^n = (a \times b)^n$

$= 10^3$

$= \mathbf{1000}$

(c) $(\sqrt[3]{3})^6 \times (\sqrt{3})^{-1} \div (\sqrt[4]{3})^{10} = \left(3^{\frac{1}{3}}\right)^6 \times \left(3^{\frac{1}{2}}\right)^{-1} \div \left(3^{\frac{1}{4}}\right)^{10}$ ⟵ $(\sqrt[n]{a})^m = a^{\frac{m}{n}}$

$= 3^2 \times 3^{-\frac{1}{2}} \div 3^{\frac{5}{2}}$ ⟵ $(a^m)^n = a^{mn}$

$= 3^{2-\frac{1}{2}-\frac{5}{2}}$ ⟵ $a^m \times a^n = a^{m+n}$ and $a^m \div a^n = a^{m-n}$

$= 3^{-1}$

$= \mathbf{\frac{1}{3}}$ ⟵ $a^{-n} = \frac{1}{a^n}$

(d) $\left[(\sqrt[3]{49})^4 \div \left(\frac{1}{343}\right)^{-\frac{2}{3}}\right] \div 7^{-\frac{1}{3}} = \left[(\sqrt[3]{7^2})^4 \div \left(\frac{1}{7^3}\right)^{-\frac{2}{3}}\right] \div \frac{1}{7^{\frac{1}{3}}}$ ⟵ $a^{-n} = \frac{1}{a^n}$

$= \left[\left(7^{\frac{2}{3}}\right)^4 \div (7^{-3})^{-\frac{2}{3}}\right] \times 7^{\frac{1}{3}}$ ⟵ $(\sqrt[n]{a})^m = a^{\frac{m}{n}}$

$= \left(7^{\frac{8}{3}} \div 7^2\right) \times 7^{\frac{1}{3}}$ ⟵ $(a^m)^n = a^{mn}$

$= 7^{\frac{2}{3}} \times 7^{\frac{1}{3}}$ ⟵ $a^m \div a^n = a^{m-n}$

$= 7^{\frac{2}{3}+\frac{1}{3}}$ ⟵ $a^m \times a^n = a^{m+n}$

$= \mathbf{7}$

TRY TUTORIAL 6.1: Question 1

WORKED EXAMPLE 2 **Simplifying exponential expressions**

Simplify $\frac{16^x + 4^{2x+1}}{8^{x-2} \div 2^{4x-6}}$ and express it in the form ka^{kx}, where a and k are integers.

SOLUTION

$$\frac{16^x + 4^{2x+1}}{8^{x-2} \div 2^{4x-6}} = \frac{(2^4)^x + (2^2)^{2x+1}}{(2^3)^{x-2} \div 2^{4x-6}} \quad \longleftarrow \quad (a^m)^n \neq a^{m+n}$$

$$= \frac{2^{4x} + 2^{4x+2}}{2^{3x-6} \div 2^{4x-6}}$$

$$= \frac{2^{4x} + (2^{4x})(2^2)}{2^{3x-6-4x+6}}$$

$$= \frac{2^{4x} + 4(2^{4x})}{2^{-x}}$$

$$= \frac{5(2^{4x})}{2^{-x}}$$

$$= 5(2^{4x-(-x)}) \quad \longleftarrow \quad \frac{a^m}{a^n} \neq a^{\frac{m}{n}}$$

$$= 5(2^{5x})$$

TRY TUTORIAL 6.1: Question 2

WORKED EXAMPLE 3 **Finding the value of an exponential expression**

Given that $5^{2x+4} \times 4^{x-1} = 10^{x+3}$, find the value of 10^x.

SOLUTION

$$5^{2x+4} \times 4^{x-1} = 10^{x+3}$$

$$5^{2x+4} \times (2^2)^{x-1} = 10^x \times 10^3$$

$$5^{2x+4} \times 2^{2x-2} = 10^x \times 10^3$$

$$5^{2x} \times 5^4 \times 2^{2x} \times \frac{1}{4} = 10^x \times 1000$$

$$\frac{5^{2x} \times 2^{2x}}{10^x} = \frac{1000 \times 4}{5^4}$$

$$\frac{10^{2x}}{10^x} = \frac{32}{5}$$

$$10^x = \frac{32}{5}$$

TRY TUTORIAL 6.1: Questions 3, 4

WORKED EXAMPLE 4 Solving equations involving exponential expressions

Solve each of the following equations.

(a) $7^{x^2-4} = 1$

(b) $27^{2x} - 9^{5x+1} = 0$

(c) $11^x \times 2^{3x+1} = \frac{1}{44}$

(d) $(\sqrt{5})^{2x-4} + \left(\frac{1}{\sqrt[3]{5}}\right)^{-3x} = 26$

SOLUTION

(a)
$$7^{x^2-4} = 1$$
$$7^{x^2-4} = 7^0$$
$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

(b)
$$27^{2x} - 9^{5x+1} = 0$$
$$27^{2x} = 9^{5x+1}$$
$$(3^3)^{2x} = (3^2)^{5x+1}$$
$$3^{6x} = 3^{10x+2}$$
$$6x = 10x + 2$$
$$4x = -2$$
$$x = -\frac{1}{2}$$

(c)
$$11^x \times 2^{3x+1} = \frac{1}{44}$$
$$11^x \times 2^{3x} \times 2 = \frac{1}{44}$$
$$11^x \times (2^3)^x = \frac{1}{88}$$
$$11^x \times 8^x = \frac{1}{88}$$
$$88^x = 88^{-1}$$
$$x = -1$$

(d)
$$(\sqrt{5})^{2x-4} + \left(\frac{1}{\sqrt[3]{5}}\right)^{-3x} = 26$$
$$\left(5^{\frac{1}{2}}\right)^{2x-4} + \left(5^{-\frac{1}{3}}\right)^{-3x} = 26$$
$$5^{x-2} + 5^x = 26$$
$$\frac{5^x}{5^2} + 5^x = 26$$
$$\frac{1}{25}(5^x) + 5^x = 26$$
$$\frac{26}{25}(5^x) = 26$$
$$5^x = 25$$
$$5^x = 5^2$$
$$x = 2$$

TRY TUTORIAL 6.1: Question 5

WORKED EXAMPLE 5 **Solving equations involving exponential expressions and substituti**

(i) Given that $u = 2^x$, express $2^{2x+1} = 5(2^{x+2}) - 32$ as an equation in u.

(ii) Hence find the values of x for which $2^{2x+1} = 5(2^{x+2}) - 32$.

(iii) Explain why the equation $2^{2x+1} = 5(2^{x+2}) - k$ has no solution if $k > 50$.

SOLUTION

(i) $2^{2x+1} = 5(2^{x+2}) - 32$

$2^{2x} \times 2 = 5(2^x \times 2^2) - 32$

$(2^x)^2 \times 2 = 5(2^x \times 4) - 32 \quad$ — (1)

Substitute $u = 2^x$ into (1):

$2u^2 = 20u - 32$

$2u^2 - 20u + 32 = 0$

$\mathbf{u^2 - 10u + 16 = 0}$

(ii) $2^{2x+1} = 5(2^{x+2}) - 32$

From (i),

$u^2 - 10u + 16 = 0$

$(u - 2)(u - 8) = 0$

$u = 2 \quad$ or $\quad u = 8$

When $u = 2$, i.e. $2^x = 2$,

$x = 1$

When $u = 8$, i.e. $2^x = 8$,

$2^x = 2^3$

$x = 3$

$\therefore$ $\mathbf{x = 1}$ or $\mathbf{x = 3}$

(iii) $2^{2x+1} = 5(2^{x+2}) - k$

$2^{2x} \times 2 = 5(2^x \times 2^2) - k$

$(2^x)^2 \times 2 = 5(2^x \times 4) - k$

Substitute $u = 2^x$:

$2u^2 = 20u - k$

$2u^2 - 20u + k = 0$

Discriminant $= (-20)^2 - 4(2)(k)$ ⟵ $a = 2, b = -20, c = k$

$= 400 - 8k$

When the equation has no solution,

$$\text{Discriminant} < 0$$
$$400 - 8k < 0$$
$$8k > 400$$
$$k > 50$$

$\therefore$ The equation has no solution if $k > 50$.

TRY TUTORIAL 6.1: Questions 6–8

WORKED EXAMPLE 6 **Solving simultaneous equations involving exponential expressions**

Find the values of x and y which satisfy the equations

$$5^{x+y} = \sqrt[3]{125},$$
$$\frac{3^y}{9^x} = \left(\frac{1}{3}\right)^{-4}.$$

SOLUTION

$5^{x+y} = \sqrt[3]{125}$ — (1)

$\frac{3^y}{9^x} = \left(\frac{1}{3}\right)^{-4}$ — (2)

From (1),

$5^{x+y} = 5$

$x + y = 1$ — (3)

From (2),

$\frac{3^y}{3^{2x}} = 3^4$

$3^{y-2x} = 3^4$

$y - 2x = 4$ — (4)

(3) – (4): $3x = -3$

$x = -1$

Substitute $x = -1$ into (3):

$-1 + y = 1$

$y = 2$

$\therefore$ $\boldsymbol{x = -1, y = 2}$

TRY TUTORIAL 6.1: Questions 9–11

NAME: ____________________ CLASS: ____________ DATE: ________

TUTORIAL 6.1

1. Without using a calculator, simplify each of the following.

   (a) $45^{\frac{1}{2}} \times 45^{-\frac{5}{6}} \div 45^{\frac{2}{3}}$

   (b) $(6^5 \div 24^6) \times 2^{10}$

   (c) $\dfrac{(\sqrt{8})^3 \times (\sqrt[3]{8})^2}{\sqrt{2}}$

   (d) $\sqrt[3]{81} \times \dfrac{1}{9^{-\frac{1}{2}}} \div \sqrt[3]{3} - 3^2$

   (e) $49^{\frac{1}{2}} \times 49^{\frac{1}{x}} \div 343^{x+\frac{1}{3}}$

   (f) $\left(\dfrac{1}{\sqrt[3]{121^x}} \div \dfrac{1}{\sqrt{121^{1-x}}}\right)^6$

2. Simplify $\dfrac{25^{2x-1}+5^{4x+1}}{(5^{3x-1})^2}$ and express it in the form ka^{-2x}, where a and k are integers.

3. Without using a calculator, find the value of 10 000^x, given that $5^{x+2} = 20^{3-x}$.

4. Given that $\sqrt{343^x} = \dfrac{7^{1-x}}{49}$, find the value of $\sqrt{343^x}$.

5. Solve each of the following equations.

   (a) $27^x = \dfrac{1}{9}$

   (b) $11^{1.5x} = \sqrt[4]{1331}$

   (c) $6^{x^2-6} = 216$

   (d) $(49^x)^x = 7^{x+3}$

   (e) $5^{2x} \times 4^x = \dfrac{1}{\sqrt{10}}$

   (f) $32^{\frac{1}{x}} = \sqrt{8\sqrt{8}}$

6. Using the substitution $u = 3^x$, solve the equation $3^{x+1} + 3^{-x} = 4$.

7. Using a suitable substitution, or otherwise, solve each of the following equations.

   (a) $6^x - 6^{1-x} = 5$

   (b) $2^{x+2} - 2^{x+1} = 128$

8. (i) Given that $u = 3^x$, express $3^{2x+1} = 4(3^{x+2}) - 3^4$ as an equation in u.

   (ii) Hence find the values of x for which $3^{2x+1} = 4(3^{x+2}) - 3^4$.

   (iii) Explain why the equation $3^{2x+1} = 4(3^{x+2}) - k$ has real solutions only if $k \leqslant 108$.

9. Find the values of x and y which satisfy the equations $\sqrt{5^{x+1}} = \dfrac{125^{x+2y}}{5} = 5^y$.

10. Without using a calculator, solve, for x and y, the simultaneous equations

$$5^x \times 125^y = 1,$$
$$81^{x-4} \div 3^y = 27^{\frac{1}{y}}.$$

11. Find the values of x and y which satisfy the equations

$$4^x - 9^y = 13,$$
$$4^{x-1} + 9^{y+1} = 31.$$

6.2 Introduction to logarithms

Objectives Checklist

- Simplify expressions involving logarithms

Notes and Worked Examples

The exponential form (or index form) $y = a^x$ is **equivalent** to the logarithmic form $x = \log_a y$, where $a > 0, a \neq 1$.

index

$$y = a^x \Leftrightarrow x = \log_a y$$

base

For $\log_a y$ to be defined, $a > 0$, $a \neq 1$ and $y > 0$.

Converting between the index form and the logarithmic form is useful in solving equations:

Index form ⇔ Common logarithmic form	Index form ⇔ Natural logarithmic form
$y = 10^x \Leftrightarrow x = \lg y$	$y = e^x \Leftrightarrow x = \ln y$

For any $a > 0$, $a \neq 1$,

Special Property 1: $\log_a a = 1$

Special Property 2: $\log_a 1 = 0$

In particular, $\lg 10 = 1$ and $\lg 1 = 0$; $\ln e = 1$ and $\ln 1 = 0$.

$\log_{10}$ is the **common logarithm** and is denoted by lg.

$\log_e$ is the **natural logarithm** and is denoted by ln.

WORKED EXAMPLE 1 **Determining whether a logarithm is defined**

(i) Determine whether $\log_x (8 - x)$ is defined for each of the following values of x.

(a) 3 (b) 1

(c) $-\frac{1}{2}$ (d) 24

(ii) Find the range of values of x for which $\log_x (8 - x)$ is defined.

SOLUTION

(i) (a) When $x = 3$, we have $\log_3 5$. $5 > 0$ and $3 > 0$

$\therefore \log_3 5$ is defined.

(b) When $x = 1$, we have $\log_1 7$.

Since the base is 1, $\log_1 7$ is not defined.

(c) When $x = -\frac{1}{2}$, we have $\log_{-\frac{1}{2}} \frac{17}{2}$.

Since the base is $-\frac{1}{2} < 0$, $\log_{-\frac{1}{2}} \frac{17}{2}$ is not defined.

(d) When $x = 24$, we have $\log_{24} (-16)$.

Since $-16 < 0$, $\log_{24} (-16)$ is not defined.

(ii) For $\log_x (8 - x)$ to be defined,

$x > 0$, $\quad 8 - x > 0 \quad$ and $\quad x \neq 1$.

$\qquad\qquad x < 8$

$\therefore$ **$0 < x < 8$, $x \neq 1$**

TRY TUTORIAL 6.2: Questions 1, 2

WORKED EXAMPLE 2 Converting index form into logarithmic form

Convert each of the following into logarithmic form.

(a) $3^4 = 81$

(b) $5^{-2} = \frac{1}{25}$

(c) $10^a = 60$

(d) $b^n = 7.9$

SOLUTION

(a) $3^4 = 81$

$\mathbf{4 = \log_3 81}$

(b) $5^{-2} = \frac{1}{25}$

$\mathbf{-2 = \log_5 \frac{1}{25}}$

(c) $10^a = 60$

$\mathbf{a = \log_{10} 60}$

(d) $b^n = 7.9$

$\mathbf{n = \log_b 7.9}$

TRY TUTORIAL 6.2: Question 3

WORKED EXAMPLE 3 Converting logarithmic form into index form

Convert each of the following into index form.

(a) $\log_2 128 = 7$

(b) $\log_7 \frac{1}{343} = -3$

(c) $a = \log_9 5$

(d) $n = \log_b 2.6$

SOLUTION

(a) $\log_2 128 = 7$

$\mathbf{2^7 = 128}$

(b) $\log_7 \frac{1}{343} = -3$

$\mathbf{7^{-3} = \frac{1}{343}}$

(c) $a = \log_9 5$

$\mathbf{9^a = 5}$

(d) $n = \log_b 2.6$

$\mathbf{b^n = 2.6}$

TRY TUTORIAL 6.2: Questions 4–6

WORKED EXAMPLE 4 Finding an expression for a logarithmic term

Given that $\log_5 a = m$ and $\log_{125} b = n$, find an expression, in terms of m and n, for ab.

SOLUTION

From $\log_5 a = m$, we have $a = 5^m$.

From $\log_{125} b = n$, we have $b = 125^n = 5^{3n}$.

$\therefore ab = 5^m \times 5^{3n} = \mathbf{5^{m+3n}}$

TRY TUTORIAL 6.2: Questions 7, 8

WORKED EXAMPLE 5 Evaluating common logarithms and natural logarithms

Use a calculator to evaluate each of the following.

(a) lg 61

(b) lg 0.61

(c) ln 4e

(d) ln 4 + ln e

SOLUTION

(a) lg 61 = **1.79** (to 3 s.f.)

(b) lg 0.61 = **−0.215** (to 3 s.f.)

(c) ln 4e = **2.39** (to 3 s.f.)

(d) ln 4 + ln e = **2.39** (to 3 s.f.)

TRY TUTORIAL 6.2: Question 9

WORKED EXAMPLE 6 Applying special properties of logarithms

(a) Without using a calculator, find the value of $\log_7 7 - 5 \log_{8.4} 1$.

(b) Given that $p > 1$, evaluate $(\log_p p)^6 \times (3 \ln e)$.

SOLUTION

(a) $\log_7 7 - 5 \log_{8.4} 1 = 1 - 5(0)$

$= \mathbf{1}$

(b) $(\log_p p)^6 \times (3 \ln e) = 1^6 \times (3 \times 1)$

$= \mathbf{3}$

TRY TUTORIAL 6.2: Question 10

WORKED EXAMPLE 7 Solving simple logarithmic equations

Solve each of the following equations.

(a) $\log_9 x = 2$

(b) $\ln 5x = 1.6$

(c) $\log_x 8 = 3$

(d) $\log_x (8x - 7) = 2$

SOLUTION

(a) $\log_9 x = 2$

$x = 9^2$

$= \mathbf{81}$

(b) $\ln 5x = 1.6$

$5x = e^{1.6}$

$x = \frac{1}{5}e^{1.6}$

$= \mathbf{0.991}$ (to 3 s.f.)

(c) $\log_x 8 = 3$

$x^3 = 8$

$x = \sqrt[3]{8}$

$= \mathbf{2}$

(d) $\log_x (8x - 7) = 2$

$x^2 = 8x - 7$

$x^2 - 8x + 7 = 0$

$(x - 1)(x - 7) = 0$

$x = 1$ or $x = 7$

(rejected)

$\therefore \mathbf{x = 7}$ ← The logarithm is not defined when $x = 1$.

TRY TUTORIAL 6.2: Question 11

IDEAS

NAME: ______________________ CLASS: __________ DATE: ______

TUTORIAL **6.2**

1. Find the range of values of x for which $\log_x (x - 4)$ is defined.

2. Suggest a value of x for which $\log_{x-5} (19 - 3x)$ is defined.

3. Convert each of the following into the logarithmic form.

   (a) $5^3 = 125$ (b) $7^{-2} = \frac{1}{49}$

   (c) $8^p = 0.9$ (d) $a^q = 6$

4. Convert each of the following into the index form.

   (a) $\log_3 243 = 5$ (b) $\log_{11} \frac{1}{1331} = -3$

   (c) $x = \log_6 7$ (d) $z = \log_y 9.08$

5. Given that $10^{2y} - 3 = x$, find an expression for y in terms of x.

6. It is given that $\ln (5x - 2y) = k + 6$, where k is a constant. Express y in terms of x.

7. Given that $\log_4 a = m$ and $\log_8 b = n$, find an expression, in terms of m and n, for each of the following.

   (a) a^2b (b) $\frac{a}{\sqrt[3]{b}}$

8. It is given that $\log_{81} a = m$, $\log_{27} b = n$ and $\frac{a}{b} = 9^k$. Express k in terms of m and n.

9. Use a calculator to evaluate each of the following.

   (a) $\lg 30$ (b) $\lg\frac{4}{7}$

   (c) $\ln\frac{4}{7}$ (d) $\ln e - \ln 2.6$

10. Without using a calculator, simplify each of the following.

    (a) $\log_6 6 - 249 \log_8 1 + 3 \lg 10$ (b) $\frac{2}{3}\ln e \times \left(\log_{5.5} 5.5\right)^{5.5}$

    (c) $(5 - 3 \log_3 3)^7$ (d) $\frac{(\ln 1 + \lg 1)^{88}}{88}$

    (e) $\log_4 (21 - 20 \ln e)$ (f) $\sqrt{\left(3\log_7 7\right)^2 + 16(\lg 10)^4}$

11. Solve each of the following equations.

(a) $\log_2 x = 6$

(b) $\ln x = -0.5$

(c) $\log_9 (7x + 9) = 1$

(d) $\log_x 625 = 4$

(e) $\log_x (5x - 4) = 2$

(f) $3(\log_8 x)^2 = \log_8 x$

(g) $\lg (\ln x) = 0$

(h) $\ln (\lg x) = \frac{4}{5}$

6.3 Laws of Logarithms and Change of Base Formula

Objectives Checklist

- Simplify expressions involving logarithms

Notes and Worked Examples

If a, x and y are positive numbers, $a \neq 1$ and r is a real number, then:

Product Law of Logarithms: $\log_a x + \log_a y = \log_a xy$

Quotient Law of Logarithms: $\log_a x - \log_a y = \log_a \frac{x}{y}$

Power Law of Logarithms: $\log_a x^r = r \log_a x$

If a, b and c are positive numbers, $a \neq 1$ and $c \neq 1$, then:

Change of Base Formula: $\log_a b = \dfrac{\log_c b}{\log_c a} = \dfrac{1}{\log_b a}$

For any $a > 0$, $a \neq 1$,

Special Property 3: $a^{\log_a b} = b$

WORKED EXAMPLE 1 **Applying the Product Law**

(a) Express each of the following as a single logarithm.

(i) $\log_8 24 + \log_8 6$ **(ii)** $\log_a 25 + \log_a 1.2 + \log_a 10$

(b) Simplify each of the following.

(i) $\log_{15} 3 + \log_{15} 5$ **(ii)** $\log_3 2.5 + \log_3 0.2 + \log_3 2$

SOLUTION

(a) (i) $\log_8 24 + \log_8 6 = \log_8 (24 \times 6)$ ⟵ $\log_a x + \log_a y = \log_a xy$

$= \mathbf{\log_8 144}$

(ii) $\log_a 25 + \log_a 1.2 + \log_a 10 = \log_a (25 \times 1.2 \times 10)$ ⟵ $\log_a x + \log_a y = \log_a xy$

$= \mathbf{\log_a 300}$

(b) (i) $\log_{15} 3 + \log_{15} 5 = \log_{15} (3 \times 5)$ ⟵ $\log_a x + \log_a y = \log_a xy$

$= \log_{15} 15$

$= \mathbf{1}$

(ii) $\log_3 2.5 + \log_3 0.2 + \log_3 2 = \log_3 (2.5 \times 0.2 \times 2)$ ⟵ $\log_a x + \log_a y = \log_a xy$

$= \log_3 1$

$= \mathbf{0}$

TRY TUTORIAL 6.3: Questions 1, 2

WORKED EXAMPLE 2 Applying the Quotient Law

(a) Express each of the following as a single logarithm.

(i) $\log_5 81 - \log_5 27$ **(ii)** $(\log_a 60 - \log_a 6) - \log_a 2$

(b) Simplify each of the following.

(i) $\log_2 126 - \log_2 63$ **(ii)** $\log_{4.8} 50 - (\log_{4.8} 100 - \log_{4.8} 2)$

SOLUTION

(a) (i) $\log_5 81 - \log_5 27 = \log_5 \frac{81}{27}$ ⟵ $\log_a x - \log_a y = \log_a \frac{x}{y}$

$= \mathbf{\log_5 3}$

(ii) $(\log_a 60 - \log_a 6) - \log_a 2 = \log_a \frac{60}{6} - \log_a 2$ ⟵ $\log_a x - \log_a y = \log_a \frac{x}{y}$

$= \log_a 10 - \log_a 2$

$= \log_a \frac{10}{2}$

$= \mathbf{\log_a 5}$

(b) (i) $\log_2 126 - \log_2 63 = \log_2 \frac{126}{63}$ ⟵ $\log_a x - \log_a y = \log_a \frac{x}{y}$

$= \log_2 2$

$= \mathbf{1}$

(ii) $\log_{4.8} 50 - (\log_{4.8} 100 - \log_{4.8} 2) = \log_{4.8} 50 - \log_{4.8} \frac{100}{2}$ ⟵ $\log_a x - \log_a y = \log_a \frac{x}{y}$

$= \log_{4.8} 50 - \log_{4.8} 50$

$= \mathbf{0}$

TRY TUTORIAL 6.3: Questions 3–5

WORKED EXAMPLE 3 Applying the Power Law

Find the value of each of the following.

(a) $\log_5 \sqrt{125}$ **(b)** $\sqrt[3]{\log_2 256}$

SOLUTION

(a) $\log_5 \sqrt{125} = \log_5 5^{\frac{3}{2}}$

$= \frac{3}{2}\log_5 5 \longleftarrow \log_a x^r = r\log_a x$

$= \frac{3}{2}$

(b) $\sqrt[3]{\log_2 256} = \sqrt[3]{\log_2 2^8}$

$= \sqrt[3]{8\log_2 2} \longleftarrow \log_a x^r = r\log_a x$

$= \sqrt[3]{8}$

$= 2$

TRY TUTORIAL 6.3: Question 6

WORKED EXAMPLE 4 Simplifying logarithms using the Laws of Logarithms

Express $3 + \log_6 10$ as a single logarithmic term.

SOLUTION

$3 + \log_6 10 = 3\log_6 6 + \log_6 10$

$= \log_6 6^3 + \log_6 10 \longleftarrow r\log_a x = \log_a x^r$

$= \log_6 216 + \log_6 10$

$= \mathbf{\log_6 2160} \longleftarrow \log_a x + \log_a y = \log_a xy$

TRY TUTORIAL 6.3: Question 7

WORKED EXAMPLE 5 Applying the Change of Base Formula

Use a calculator to evaluate each of the following.

(a) $\log_2 9$ **(b)** $\log_7 3.5$

SOLUTION

(a) $\log_2 9 = \frac{\lg 9}{\lg 2}$ ⟵ or $\frac{\ln 9}{\ln 2}$

$= \mathbf{3.17}$ (to 3 s.f.)

(b) $\log_7 3.5 = \frac{\lg 3.5}{\lg 7}$ ⟵ or $\frac{\ln 3.5}{\ln 7}$

$= \mathbf{0.644}$ (to 3 s.f.)

TRY TUTORIAL 6.3: Question 8

WORKED EXAMPLE 6 Simplifying logarithms using the Change of Base Formula

Without using a calculator, show that $\log_9 8 \times \log_2 7 \times \log_{\sqrt{7}} 9 = 6$.

SOLUTION

$$\log_9 8 \times \log_2 7 \times \log_{\sqrt{7}} 9 = \frac{\lg 8}{\lg 9} \times \frac{\lg 7}{\lg 2} \times \frac{\lg 9}{\lg \sqrt{7}}$$

$$= \frac{\lg 2^3}{\lg 9} \times \frac{\lg 7}{\lg 2} \times \frac{\lg 9}{\lg 7^{\frac{1}{2}}}$$

$$= \frac{3\lg 2}{\lg 9} \times \frac{\lg 7}{\lg 2} \times \frac{\lg 9}{\frac{1}{2}\lg 7}$$

$$= 3 \times \frac{1}{\frac{1}{2}}$$

$$= 6 \text{ (shown)}$$

TRY TUTORIAL 6.3: Questions 9–11

WORKED EXAMPLE 7 Finding expressions for logarithmic terms

Given that $\log_5 2 = p$ and $\log_5 7 = q$, find an expression, in terms of p and q, for each of the following.

(a) $\log_5 14$

(b) $\log_5 0.7$

SOLUTION

(a)
$$\begin{aligned}\log_5 14 &= \log_5 (2 \times 7)\\ &= \log_5 2 + \log_5 7\\ &= \boldsymbol{p + q}\end{aligned}$$

(b)
$$\begin{aligned}\log_5 0.7 &= \log_5 \frac{7}{10}\\ &= \log_5 \frac{7}{2\times 5}\\ &= \log_5 7 - (\log_5 2 + \log_5 5)\\ &= q - (p + 1)\\ &= \boldsymbol{q - p - 1}\end{aligned}$$

TRY TUTORIAL 6.3: Questions 12, 13

NAME: ____________________ CLASS: ____________ DATE: ________

TUTORIAL 6.3

1. Express each of the following as a single logarithm.

   (a) $\lg 12 + \lg 5$ (b) $\log_3 8 + \log_3 7$

2. Find the value of each of the following.

   (a) $\log_4 2.5 + \log_4 25.6$ (b) $\ln e^2 + \ln e^{-1}$

3. Express each of the following as a single logarithm.

   (a) $\ln 28 - \ln 4$ (b) $\log_5 9 - \log_5 0.6$

4. Find the value of each of the following.

   (a) $-\lg 10 + \lg 4 + \lg 25$ (b) $\log_{\sqrt{5}} 24 - \left(\log_{\sqrt{5}} 3 + \log_{\sqrt{5}} 8\right)$

5. Given that $\ln x^3y = p$ and $\ln xy^3 = q$, find an expression, in terms of p and q, for each of the following.

   (a) $\ln xy$ (b) $\ln \dfrac{ex}{y}$

6. Find the value of each of the following.

   (a) $\log_5 \dfrac{1}{25}$ (b) $\log_4 \sqrt[3]{16}$

   (c) $\log_7 49^{24}$ (d) $\sqrt{\log_3 81}$

7. Express each of the following as a single logarithmic term.

   (a) $1 + \lg 7$ (b) $\log_2 10 - 3$

   (c) $\dfrac{1}{2}\lg a^6 + 5$ (d) $\ln b^2\sqrt{b} - 4\log_5 25$

8. Use a calculator to evaluate each of the following.

   (a) $\log_5 9$ (b) $\log_{2.4} 71$

9. Without using a calculator, show that $2 + \dfrac{1}{\log_3 2} + \dfrac{\ln 7}{\ln 2} = \log_2 84$.

10. Without using a calculator, show that $\log_5 8 \times \log_{128} 3 \times \log_{81} 125 = \dfrac{9}{28}$.

11. Given that $\log_4 x = p$, express each of the following in terms of p.

(a) $\log_4 4x^2$

(b) $\log_4 \frac{16}{x}$

(c) $(\log_2 2x)^2$

(d) $5\sqrt{x}$

12. It is given that $\log_2 3 = 1.58$ and $\log_2 25 = 4.64$. Find the value of each of the following.

(a) $\log_2 6$

(b) $\log_2 0.12$

(c) $\log_2 15$

(d) $\log_2 \sqrt{\frac{25}{27}}$

13. Given that $\log_3 4 = m$ and $\log_3 5 = n$, find an expression, in terms of m and/or n, for each of the following.

(a) $\log_3 20$

(b) $\log_3 3.2$

(c) $\log_3 3.75$

(d) $\lg 3$

6.4 Logarithmic and exponential equations

Objectives Checklist

- Solve equations involving exponents and logarithms

Notes and Worked Examples

To solve an equation involving logarithms:

Case	Condition	Method
1	The equation simplifies to $\log_a p = \log_a q$.	Apply the **Equality of Logarithms**: $\log_a p = \log_a q \Leftrightarrow p = q$
2	The equation simplifies to $\log_a x = b$.	Convert it to the exponential form: $\log_a x = b \Leftrightarrow x = a^b$

To solve an exponential equation $a^x = b$:

Case	Condition	Method
1	b can be expressed as a^y	$a^x = a^y \Rightarrow x = y$
2	b cannot be expressed as a^y	Take the logarithm (lg or ln) on both sides of the equation to get $x \lg a = \lg b \Rightarrow x = \frac{\lg b}{\lg a}$.

Solving an exponential equation $p(a^{2x}) + q(a^x) + r = 0$ requires substitution and factorisation:

Step 1: Let $u = a^x$ to get a quadratic equation in u.

Step 2: Factorise the expression on the LHS and solve for the value(s) of u.

Step 3: Replace u with a^x to solve for the value(s) of x.

When we solve logarithmic and exponential equations, extraneous solutions may be introduced, so always check whether any solution has to be rejected.

WORKED EXAMPLE 1 **Solving logarithmic equations involving the Laws of Logarithms**

Solve each of the following equations.

(a) $\log_{12}(x-4)+\log_{12}(x-5)=1$ **(b)** $2\log_x 12-\frac{1}{2}\log_x 16=2$

SOLUTION

(a) $\log_{12}(x-4)+\log_{12}(x-5)=1$

Method 1:

$$\log_{12}[(x-4)(x-5)]=\log_{12}12$$
$$(x-4)(x-5)=12$$
$$x^2-9x+20=12$$
$$x^2-9x+8=0$$
$$(x-1)(x-8)=0$$
$$x=1 \quad \text{or} \quad x=8$$

Method 2:

$$\log_{12}[(x-4)(x-5)]=1$$
$$(x-4)(x-5)=12^1$$
$$x^2-9x+20=12$$
$$x^2-9x+8=0$$
$$(x-1)(x-8)=0$$
$$x=1 \quad \text{or} \quad x=8$$

Check: When $x=1$, $\log_{12}(1-4)=\log_{12}(-3)$ and $\log_{12}(1-5)=\log_{12}(-4)$ are not defined, i.e. $x=1$ is rejected.

Check: When $x=8$, $\log_{12}(8-4)=\log_{12}4$ and $\log_{12}(8-5)=\log_{12}3$ are both defined.

$\therefore$ $x=\mathbf{8}$

(b) $2\log_x 12-\frac{1}{2}\log_x 16=2$

Method 1:

$$\log_x 12^2-\log_x 16^{\frac{1}{2}}=2\log_x x$$
$$\log_x 144-\log_x 4=\log_x x^2$$
$$\log_x \frac{144}{4}=\log_x x^2$$
$$\log_x 36=\log_x x^2$$
$$x^2=36$$
$$x=\pm 6$$

Method 2:

$$\log_x 12^2-\log_x 16^{\frac{1}{2}}=2$$
$$\log_x 144-\log_x 4=2$$
$$\log_x \frac{144}{4}=2$$
$$\log_x 36=2$$
$$x^2=36$$
$$x=\pm 6$$

Check: When $x=6$, $\log_6 12$ and $\log_6 16$ are both defined.

Check: When $x=-6$, $\log_{(-6)}12$ and $\log_{(-6)}16$ are not defined, i.e. $x=-6$ is rejected.

$\therefore$ $x=\mathbf{6}$

TRY TUTORIAL 6.4: Questions 1–5

WORKED EXAMPLE 2 **Solving logarithmic equations involving the Change of Base Formula**

Solve each of the following equations.

(a) $\log_3 (2x + 15) = \dfrac{2}{\log_x 3}$ **(b)** $\log_2 x = 1 + 6 \log_x 2$

SOLUTION

(a) $\log_3 (2x + 15) = \dfrac{2}{\log_x 3}$

$\log_3 (2x + 15) = 2 \log_3 x$

$\log_3 (2x + 15) = \log_3 x^2$

$2x + 15 = x^2$

$x^2 - 2x - 15 = 0$

$(x - 5)(x + 3) = 0$

$x = 5$ or $x = -3$

Check: When $x = 5$, $\log_3 [2(5) + 15] = \log_3 25$ and $\log_5 3$ are defined.

Check: When $x = -3$, $\log_{(-3)} 3$ is not defined, i.e. $x = -3$ is rejected.

$\therefore x = \mathbf{5}$

(b) $\log_2 x = 1 + 6 \log_x 2$

$\log_2 x = 1 + \dfrac{6}{\log_2 x}$

Let $u = \log_2 x$.

$u = 1 + \dfrac{6}{u}$

$u^2 = u + 6$

$u^2 - u - 6 = 0$

$(u - 3)(u + 2) = 0$

$u = 3$ or $u = -2$

When $u = 3$, $\log_2 x = 3$,

$x = 2^3$

$= 8$

When $u = -2$, $\log_2 x = -2$

$x = 2^{-2}$

$= \dfrac{1}{4}$

Check: When $x = 8$, $\log_2 8$ and $\log_8 2$ are defined.

Check: When $x = \dfrac{1}{4}$, $\log_2 \dfrac{1}{4}$ and $\log_{\frac{1}{4}} 2$ are defined.

$\therefore x = \mathbf{8}$ or $x = \mathbf{\dfrac{1}{4}}$

TRY TUTORIAL 6.4: Questions 6–12

WORKED EXAMPLE 3 **Solving exponential equations of the form $a^x = b$**

Solve each of the following equations.

(a) $7^{x-2} = 28$

(b) $e^{0.5x} = 4$

SOLUTION

(a)

$$7^{x-2} = 28$$
$$\lg 7^{x-2} = \lg 28$$
$$(x-2)\lg 7 = \lg 28$$
$$x - 2 = \frac{\lg 28}{\lg 7}$$
$$x = \frac{\lg 28}{\lg 7} + 2$$
$$= \mathbf{3.71} \text{ (to 3 s.f.)}$$

(b)

$$e^{0.5x} = 4$$
$$\ln e^{0.5x} = \ln 4$$
$$0.5x \ln e = \ln 4$$
$$0.5x = \ln 4 \quad \leftarrow \ln e = 1$$
$$x = \frac{\ln 4}{0.5}$$
$$= \mathbf{2.77} \text{ (to 3 s.f.)}$$

TRY TUTORIAL 6.4: Question 13

WORKED EXAMPLE 4 **Solving exponential equations reducible to the form $a^x = b$**

Solve each of the following equations.

(a) $12^x = \dfrac{4^{2x+3}}{\sqrt{25^x}}$ **(b)** $e^{2x} + e^x - 2 = 0$

SOLUTION

(a)

$$12^x = \frac{4^{2x+3}}{\sqrt{25^x}}$$

$$= \frac{4^{2x} \times 4^3}{(\sqrt{25})^x}$$

$$= \frac{16^x \times 64}{5^x}$$

$$= 64 \times \left(\frac{16}{5}\right)^x$$

$$\frac{12^x}{\left(\frac{16}{5}\right)^x} = 64$$

$$\left(\frac{15}{4}\right)^x = 64$$

$$\lg\left(\frac{15}{4}\right)^x = \lg 64$$ ⟵ We can also introduce a "ln" on both sides of the equation.

$$x \lg\frac{15}{4} = \lg 64$$

$$x = \frac{\lg 64}{\lg \frac{15}{4}}$$

$$= \mathbf{3.15} \text{ (to 3 s.f.)}$$

(b) $e^{2x} + e^x - 2 = 0$

Let $u = e^x$.

$$u^2 + u - 2 = 0$$

$$(u + 2)(u - 1) = 0$$

$$u = -2 \quad \text{or} \quad u = 1$$

When $u = -2$, $e^x = -2$ (no solution).

When $u = 1$, $e^x = 1$,

$$x = 0$$ ⟵ $a^0 = 1$ provided $a \neq 0$

$\therefore x = \mathbf{0}$

TRY TUTORIAL 6.4: Questions 14–16

WORKED EXAMPLE 5 Solving simultaneous equations involving logarithmic expressions

Solve the simultaneous equations

$$\frac{\sqrt{3^x}}{9^y} = 27^{-1},$$
$$\log_2 \sqrt{y} - \log_4 (x-3) - 1 = 0.$$

SOLUTION

$\frac{\sqrt{3^x}}{9^y} = 27^{-1}$ — (1)

$\log_2 \sqrt{y} - \log_4 (x-3) - 1 = 0$ — (2)

From (1),

$$\frac{3^{\frac{1}{2}x}}{3^{2y}} = 3^{-3}$$
$$3^{\frac{1}{2}x-2y} = 3^{-3}$$
$$\frac{1}{2}x - 2y = -3$$
$$x - 4y = -6 \quad \text{— (3)}$$

From (2),

$$\log_2 y^{\frac{1}{2}} - \frac{\log_2 (x-3)}{\log_2 4} = 1$$
$$\frac{1}{2}\log_2 y - \frac{\log_2 (x-3)}{\log_2 2^2} = 1$$
$$\frac{1}{2}\log_2 y - \frac{1}{2}\log_2 (x-3) = 1$$
$$\log_2 y - \log_2 (x-3) = 2$$
$$\log_2 \frac{y}{x-3} = 2$$
$$\frac{y}{x-3} = 2^2$$
$$y = 4x - 12 \quad \text{— (4)}$$

Substitute (4) into (3):

$$x - 4(4x - 12) = -6$$
$$x - 16x + 48 = -6$$
$$-15x = -54$$
$$x = \frac{18}{5}$$

Substitute $x = \frac{18}{5}$ into (4):

$$y = 4\left(\frac{18}{5}\right) - 12$$
$$= \frac{12}{5}$$

$\therefore x = \frac{18}{5}, y = \frac{12}{5}$

TRY TUTORIAL 6.4: Questions 17, 18

NAME: ______________________ CLASS: ____________ DATE: ________

TUTORIAL **6.4**

1. Solve each of the following equations.
   (a) $\log_2 x = 9$
   (b) $\lg x = 0.8$
   (c) $\ln 3x = 6$
   (d) $\ln x = 3 \lg 4$
   (e) $\log_x 64 = 2$
   (f) $\log_{81} (x + 1) = -\frac{1}{2}$
   (g) $\ln (2x - 5) = (\lg 8)^2$
   (h) $\lg (10 - x) = \ln (e^3 + 3)$

2. Solve each of the following equations.
   (a) $\log_5 (7x + 1) = \log_5 (3x + 29)$
   (b) $\ln [2x(3x - 11)] = \ln (x - 22)$
   (c) $\log_2 x + \log_2 (4x + 7) = 1$
   (d) $\lg (x^2 - 2x - 13) - \lg (x - 5) = \lg (2x - 1)$
   (e) $2 \lg x = \lg (6x + 8) - \lg 5$
   (f) $\log_{16} (x-2)^2 + \frac{1}{2} = \log_{16} (12x - 29)$

3. Solve the equation $\log_6 (x + 7) - \log_6 (x - 7) = 1 + \log_6 \frac{1}{5}$.

4. (i) Solve the equation $\log_x 48 = 1 + \log_x (x - 2)$.
   (ii) Hence, find the value(s) of y for which $\log_{y^3} 48 = 1 + \log_{y^3} (y^3 - 2)$.

5. Given that $\lg x = \frac{1}{9 \lg x}$, find the possible values of x^3.

6. Solve each of the following equations.
   (a) $\log_3 x = 4 \log_x 3$
   (b) $(\ln x)^2 - \frac{2}{\log_x e} = 0$
   (c) $\lg x + \log_x 10 = 2$
   (d) $2 \log_2 x + 3 \log_8 x = -3$
   (e) $\log_{\sqrt{7}} (3x - 5) = \log_7 (3x^2 + 5x - 24)$
   (f) $\log_x (4x^2 - 4x + 18) \times \log_3 x = 4$

7. Solve the equation $\log_3 x + \log_{243} x = -2.4$.

8. Solve the equation $\log_y 1000 = (\lg y)^2$, giving your answer to 1 significant figure.

9. Given that $\log_2 (a^2x^2 + 6ax - 20) + 1 = \frac{1}{\log_{ax} 2}$, where $a < 0$, express x in terms of a.

10. Sam and Thomas solved the equation $\log_x 16 + \log_x 4 = 3$.
    Sam applied the Product Law of Logarithms.
    Thomas applied the Power Law of Logarithms.
    Show how each of them solved the equation.

11. **(i)** Show that $\log_7 x + \log_{49} x = \frac{3\lg x}{2\lg 7}$.

(ii) Hence solve the equation $\log_7 x + \log_{49} x = 5$.

12. Solve the equation $4 \ln 3x = 3\ln x - \frac{1}{\log_2 e}$.

13. Solve each of the following equations.

(a) $8^x = 5$

(b) $3e^x = 4.1$

(c) $10^{2-x} = 90$

(d) $6^{\frac{1}{2}x+4} = 27$

(e) $4^{\sqrt{x}} - 11 = 0$

(f) $7^{1.5x+6} - 8e = 0$

14. Solve each of the following equations.

(a) $2^{3x} \times 6^{3x} = 24$

(b) $7^{4x-5} = 9^{2x}$

(c) $17e^{x+1} \times e^{3-2x} = 45$

(d) $\frac{5^{2x}}{3^x} = 10(8^x)$

(e) $10\sqrt[3]{6^x} = 5^x$

(f) $3^{x+2} - 3^{x+1} = 3^{2-x}$

15. Carl wrote the following note.

> To solve the equation $2^{2x} + 2^x = 2$, I add the powers to get $2x + x = 1$, so $x = \frac{1}{3}$.

What misconception has Carl made? Write the correct working to solve the equation $2^{2x} + 2^x = 2$.

16. Find the value of x which satisfies the equation $e^x(2e^x - 11) = -15$.

17. Solve the simultaneous equations

$$5^x = 25^{y+1},$$

$$\log_3 (x - y) + 2 \log_9 3 = \log_3 (8 - y).$$

18. Find the values of x and y that satisfy the equations $16^x = \frac{1}{2}(4^{y+1})^2$ and $\log_x 8y - \frac{1}{\log_9 x} + 1 = 0$.

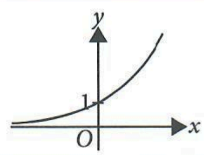
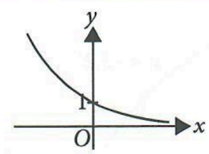
6.5 Exponential and logarithmic functions and graphs

Objectives Checklist

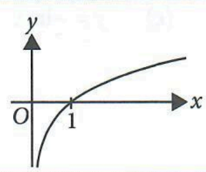
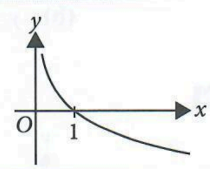
- Sketch the graphs of exponential and logarithmic functions

Notes and Worked Examples

Graphs of exponential functions of the form $y = a^x$

$y = a^x, a > 1$	$y = a^x, 0 < a < 1$
	
As $x \to \infty, y \to \infty$. As $x \to -\infty, y \to 0^+$. The graph intersects the y-axis at (0, 1). The x-axis is an asymptote.	As $x \to \infty, y \to 0^+$. As $x \to -\infty, y \to \infty$. The graph intersects the y-axis at (0, 1). The x-axis is an asymptote.

Graphs of logarithmic functions of the form $y = \log_a x$

$y = \log_a x, a > 1$	$y = \log_a x, 0 < a < 1$
	
As $x \to \infty, y \to \infty$. As $x \to 0, y \to -\infty$. The graph intersects the x-axis at (1, 0). The y-axis is an asymptote.	As $x \to \infty, y \to -\infty$. As $x \to 0, y \to \infty$. The graph intersects the x-axis at (1, 0). The y-axis is an asymptote.

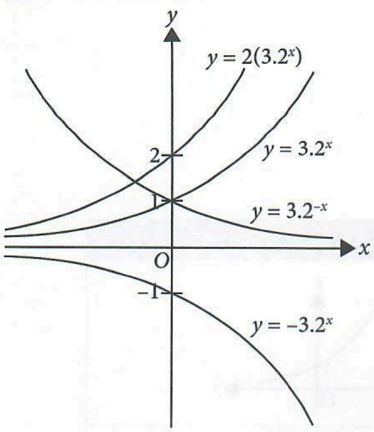
WORKED EXAMPLE 1 Sketching the graphs of exponential functions $y = a^x$

On the same diagram, sketch each of the following graphs.

(a) $y = 3.2^x$ (b) $y = 2(3.2^x)$

(c) $y = 3.2^{-x}$ (d) $y = -3.2^x$

SOLUTION



Since $a^{-x} = \frac{1}{a^x}$, then $3.2^{-x} = \frac{1}{3.2^x} = \left(\frac{5}{16}\right)^x$.

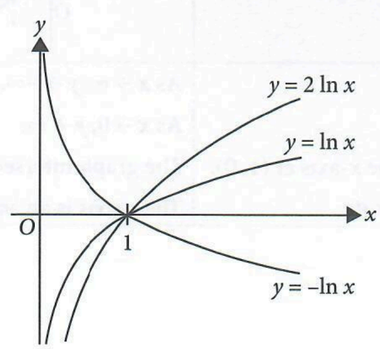
TRY TUTORIAL 6.5: Questions 1, 3, 5

WORKED EXAMPLE 2 Sketching the graphs of logarithmic functions $y = \log_a x$

On the same diagram, sketch each of the following graphs.

(a) $y = \ln x$ (b) $y = 2 \ln x$ (c) $y = -\ln x$

SOLUTION



TRY TUTORIAL 6.5: Questions 2, 4, 6

NAME: ______________________ CLASS: ____________ DATE: ________

TUTORIAL 6.5

1. Sketch the graph of each of the following functions.

(a) $y = e^x$

(b) $y = e^{-x}$

(c) $y = 6^x$

(d) $y = 4.9^x$

(e) $y = 8^{-x}$

(f) $y = \left(\frac{2}{5}\right)^x$

2. Sketch the graph of each of the following functions.

(a) $y = \ln x$

(b) $y = -\ln x$

(c) $y = \lg x$

(d) $y = \log_3 x$

(e) $y = \log_{\sqrt{7}} x$

(f) $y = \log_{0.75} x$

3. **(i)** On the same axes, sketch the graphs of $y = e^x$ and $y = 5^x$.

(ii) Use the graphs in part **(i)** to find the range of values of x for which $e^x \leqslant 5^x$.

4. **(i)** On the same axes, sketch the graphs of $y = \log_2 x$ and $y = \log_7 x$.

(ii) Use the graphs in part **(i)** to find the range of values of x for which $\log_7 x > \log_2 x$.

5. **(i)** Sketch the graph of $y = \frac{1}{4^x}$.

(ii) In order to solve the equation $x = -\log_4 (2 - 3x)$, a suitable straight line has to be drawn on the same set of axes as the graph of $y = \frac{1}{4^x}$. Find the equation of the straight line and the number of solutions.

6. The graph of $y = \log_a x$ passes through the points with coordinates $(0.25, -2)$, $(1, b)$ and $(c, 4)$.

(i) Determine the values of each of the constants a, b and c.

(ii) Hence, sketch the graph of $y = \log_a x$.

6.6 Applications of logarithmic and exponential functions

Objectives Checklist

- Model and solve problems in the sciences and in the real world using exponential and logarithmic functions

Notes and Worked Examples

Logarithmic and exponential functions can be used to model problems in the sciences and in the real world.

WORKED EXAMPLE 1 **Applying exponential functions in real-world contexts**

Cooling of pudding. A pudding is heated in an oven to a temperature of 65 °C. It subsequently cools in such a way that its temperature, T °C, t minutes after removal from the oven, is given by $T = 25 + Ae^{-kt}$, where A and k are constants.

(i) Show that $A = 40$.

When $t = 1$, the temperature of the pudding is 55 °C.

(ii) Find the value of k correct to 3 significant figures.

The pudding is to be served when its temperature is less than 45 °C.

(iii) Determine, with working, whether the pudding may be served 3 minutes after removal from the oven.

SOLUTION

(i) $T = 25 + Ae^{-kt}$

When $t = 0$, $T = 65$,

$$65 = 25 + A$$

$$A = 40 \text{ (shown)}$$

(ii) $T = 25 + Ae^{-kt}$

When $t = 1$, $T = 55$,

$$55 = 25 + 40e^{-k}$$
$$40e^{-k} = 30$$
$$e^{-k} = 0.75$$
$$-k = \ln 0.75$$
$$k = -\ln 0.75$$
$$= \mathbf{0.288} \text{ (to 3 s.f.)}$$

(iii) $T = 25 + 40e^{(\ln 0.75)t}$

When $t = 3$,

$$T = 25 + 40e^{(\ln 0.75)(3)}$$
$$= 41.875 < 45$$

Since the temperature of the pudding is less than 45 °C, the pudding may be served 3 minutes after removal from the oven.

TRY TUTORIAL 6.6: Questions 1, 2

WORKED EXAMPLE 2 **Applying logarithmic functions in real-world contexts**

Acidity. The pH value measures the acidity of a solution. If the pH of the solution is less than 7, it is acidic. If the pH is greater than 7, it is alkaline. The formula for pH value is given by the formula pH = $-\lg(H^+)$, where H^+ is the concentration, in moles per litre (mol/*l*), of hydrogen ions in the solution.

(i) Kombucha is a beverage made of fermented tea leaves. Given that the concentration of hydrogen ions in kombucha is 0.001 25 mol/ *l*, calculate its pH value, giving your answer correct to 1 decimal place.

(ii) Given that the pH value of carrot juice is 6.4, calculate its concentration of hydrogen ions. Give your answer in standard form.

(iii) A particular brand of mineral water contains half the amount of hydrogen ions as carrot juice. Is it correct to say that the mineral water has a pH of 3.2 and is acidic? Show calculations to explain your answer.

SOLUTION

(i) pH value $= -\lg 0.001\,25$

$= \mathbf{2.9}$ (to 1 d.p.)

(ii) When pH = 6.4,

$-\lg(H^+) = 6.4$

$\lg(H^+) = -6.4$

$H^+ = 10^{-6.4}$

$= 3.98 \times 10^{-7}$

∴ The concentration of hydrogen ions is $\mathbf{3.98 \times 10^{-7}}$ **mol/*l*.**

(iii) H^+ of mineral water $= \frac{1}{2} \times (3.9811 \times 10^{-7})$

$= 1.9905 \times 10^{-7}$

pH value of mineral water $= -\lg(1.9905 \times 10^{-7})$

$= 6.7$ (to 1 d.p.)

∴ The mineral water has a pH value of 6.7 instead, and it is acidic.

TRY TUTORIAL 6.6: Questions 3, 4

NAME: ______________________ CLASS: ____________ DATE: ________

TUTORIAL **6.6**

1. **Age of fossil.** The amount, A g, of carbon-14 remaining in a piece of fossilised substance is given by $A = 240e^{-kt}$, where k is a constant and t is measured in years. The carbon-14 takes approximately 5729 years to be reduced to 120 g. Calculate

(i) the value of k,

(ii) the amount of carbon-14 remaining in the substance which would indicate that it has an age of 12 000 years.

2. **Radioactivity.** Recorded values of the mass, m grams, of a radioactive substance, t hours after observations began, can be modelled by the equation $m = 72e^{-kt}$, where m_0 and k are constants. The mass of the substance is reduced to 62 grams after 2 hours.

(i) State the initial mass of the substance.

(ii) Show that $k \approx 0.0748$.

(iii) Using this value of k, find the mass of the substance after 5 hours.

3. **Intensity of earthquake.** The magnitude, M, and intensity, I, of an earthquake are connected by the formula $M = \lg\frac{I}{c}$, where c is the intensity of a 'standard' earthquake. Given that the 1938 Banda Sea earthquake registered a magnitude of 8.5 and that the 1964 Alaska earthquake was five times as strong, find, to a reasonable degree of accuracy, the magnitude of the 1964 Alaska earthquake.

4. **Safety index.** The safety index of a particular activity is defined as the logarithm of p, where the fatality rate of that activity is one in p people. A study has found that there is a one in 12 500 chance of dying from electrocution.

(i) State the safety index.

(ii) Given that the safety index of another activity is twice of the answer in part **(i)**, is it correct to conclude that there is a one in 6250 chance of a fatality resulting from this activity? Explain your answer with calculations.

NAME: ______________________ CLASS: __________ DATE: __________

Quick Test 6

[Total marks: 35]

1. Solve $(5^x)^x = 625$. [2]

2. (i) Given that $3^{2x} = 10(2^x)$, find the value of 4.5^x. [2]

   (ii) Hence, solve the equation $3^{2x} = 10(2^x)$. [2]

3. Express $2\log_4 x - \log_4(3x - 8) = \frac{1}{2}$ as a quadratic equation in x and explain why there are no real solutions. [5]

4. It is given that $\ln p - \ln 3q = \ln(p + 3q)$.

(i) Express p in terms of q. [3]

(ii) State the range of values of p and explain clearly why $0 < q < \frac{1}{3}$. [2]

5. **(i)** Given that $\log_{25} x^2 = \log_{125} u$, express u in terms of x. [3]

(ii) Find the value of x for which $\log_{125}(x^3 + 10x) - \log_{25} x^2 = \dfrac{1}{\log_3 125}$. [3]

6. **(i)** Sketch the graph of $y = \log_3 x$. [2]

(ii) In order to solve the equation $9x = \sqrt[3]{3^x}$, a suitable straight line has to be drawn on the same set of axes as the graph of $y = \log_3 x$. Find the equation of the straight line and the number of solutions. [4]

7. Find the values of x and y which satisfy the equations

$$8^{x+y} = \sqrt{\frac{1}{4}},$$

$$\log_5 (5x + y + 4) = 1.$$

[7]