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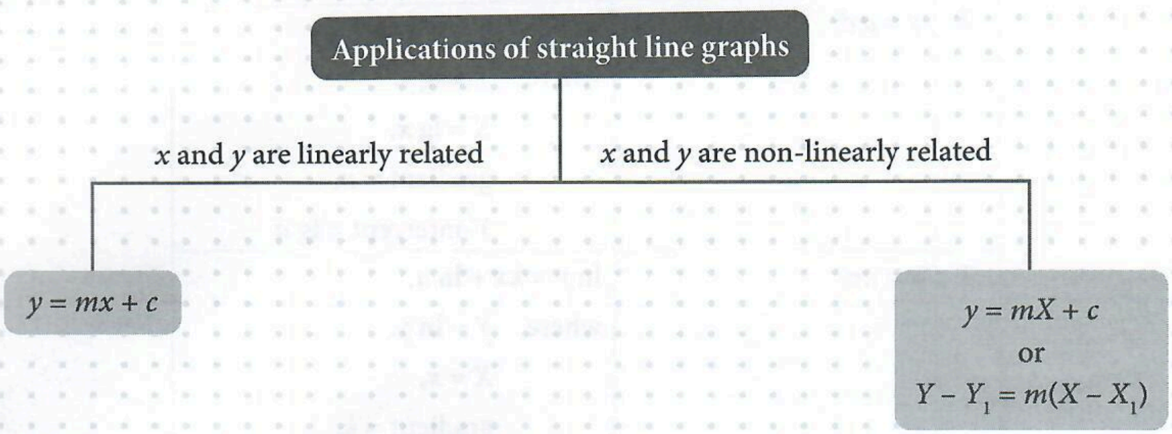
CHAPTER 8 Applications of Straight Line Graphs

Chapter Summary



Some useful conversions:

	Non-linear equation	Linear equation
1	$y = ka^x$	$\lg y = (\lg a)x + \lg k$, where $Y = \lg y$, $X = x$, gradient $= \lg a$, Y-intercept $= \lg k$
2	$y = ax^n$	$\lg y = n \lg x + \lg a$, where $Y = \lg y$, $X = \lg x$, gradient $= n$, Y-intercept $= \lg a$
3	$y = ae^{kx}$	$\ln y = kx + \ln a$, where $Y = \ln y$, $X = x$, gradient $= k$, Y-intercept $= \ln a$
4	$y = ax^2 + bx$	$\frac{y}{x} = ax + b$, where $Y = \frac{y}{x}$, $X = x$, gradient $= a$, Y-intercept $= b$ $\frac{y}{x^2} = \frac{b}{x} + a$, where $Y = \frac{y}{x^2}$, $X = \frac{1}{x}$, gradient $= b$, Y-intercept $= a$
5	$\frac{a}{x} + \frac{b}{y} = 1$	$\frac{1}{y} = -\frac{a}{b}\left(\frac{1}{x}\right) + \frac{1}{b}$, where $Y = \frac{1}{y}$, $X = \frac{1}{x}$, gradient $= -\frac{a}{b}$, Y-intercept $= \frac{1}{b}$

8.1 Converting a non-linear equation into the linear form

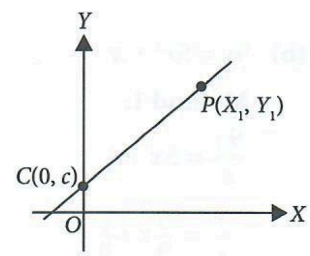
Objectives Checklist

- Convert a non-linear equation into the linear form
- Convert a linear form into a non-linear equation

Notes and Worked Examples

If a line has a gradient m and y-intercept c, then:

Equation of a non-vertical line: $y = mx + c$



When a graph of Y against X is drawn, then:

Equation of a non-vertical line: $Y = mX + c$

or

$Y - Y_1 = m(X - X_1)$

WORKED EXAMPLE 1 **Converting a non-linear equation into the linear form**

Express each of the following equations in the form $Y = mX + c$, where X and Y are functions of x and/or y, and m and c are constants. State the value of m and of c. For parts **(b)** and **(c)**, show two methods of obtaining the answer.

(a) $y = x^3 - 4$

(b) $9y = 5x^2 + x$

(c) $6x + 2y = xy$

(d) $y^2 = \dfrac{8}{\sqrt{x} - 3}$

SOLUTION

(a) $y = x^3 - 4$

$Y = mX + c$

$\therefore$ $\boldsymbol{m = 1}$ and $\boldsymbol{c = -4}$

(b) $9y = 5x^2 + x$

Method 1:

$$\frac{9y}{x} = 5x + 1$$

$$\frac{y}{x} = \frac{5}{9}x + \frac{1}{9}$$

$$Y = mX + c$$

$$\therefore \boldsymbol{m = \frac{5}{9}, c = \frac{1}{9}}$$

Method 2:

$$\frac{9y}{x^2} = 5 + \frac{1}{x}$$

$$\frac{y}{x^2} = \frac{1}{9}\left(\frac{1}{x}\right) + \frac{5}{9}$$

$$Y = mX + c$$

$$\therefore \boldsymbol{m = \frac{1}{9}, c = \frac{5}{9}}$$

(c) $6x + 2y = xy$

Method 1:

$$\frac{6x}{xy} + \frac{2y}{xy} = 1$$

$$\frac{6}{y} + \frac{2}{x} = 1$$

$$\frac{6}{y} = -\frac{2}{x} + 1$$

$$\frac{1}{y} = -\frac{1}{3x} + \frac{1}{6}$$

$$\frac{1}{y} = -\frac{1}{3}\left(\frac{1}{x}\right) + \frac{1}{6}$$

$$Y = mX + c$$

$$\therefore \boldsymbol{m = -\frac{1}{3}, c = \frac{1}{6}}$$

Method 2:

$$\frac{6x}{x} + \frac{2y}{x} = y$$

$$6 + \frac{2y}{x} = y$$

$$y = 2\left(\frac{y}{x}\right) + 6$$

$$Y = mX + c$$

$$\therefore \boldsymbol{m = 2, c = 6}$$

(d) $y^2 = \frac{8}{\sqrt{x}-3}$

$\frac{1}{y^2} = \frac{1}{8}\sqrt{x} - \frac{3}{8}$

$Y = mX + c$

$\therefore \boldsymbol{m = \frac{1}{8}}, \boldsymbol{c = -\frac{3}{8}}$

TRY TUTORIAL 8.1: Questions 1, 2

WORKED EXAMPLE 2 **Finding the values of unknowns in a non-linear equation**

The variables x and y are related by the equation $y = p\sqrt{x} + \frac{q}{\sqrt{x}}$, where p and q are constants. When a graph of $\frac{y}{\sqrt{x}}$ is plotted against $\frac{1}{x}$, the resulting line has a gradient of $-\frac{1}{3}$ and an intercept of 2 on the $\frac{y}{\sqrt{x}}$-axis. Find the value of p and of q.

SOLUTION

Method 1:

$Y = mX + c$ — (1)

Substitute $m = -\frac{1}{3}$ and $c = 2$ into (1):

$Y = -\frac{1}{3}X + 2$

Let $Y = \frac{y}{\sqrt{x}}$ and $X = \frac{1}{x}$:

$\frac{y}{\sqrt{x}} = -\frac{1}{3}\left(\frac{1}{x}\right) + 2$

$\frac{y}{\sqrt{x}} = -\frac{1}{3x} + 2$

$y = -\frac{1}{3\sqrt{x}} + 2\sqrt{x}$ ⟵ Multiply by $\sqrt{x}$ throughout.

$y = 2\sqrt{x} - \frac{1}{3\sqrt{x}}$

$\therefore \boldsymbol{p = 2}, \boldsymbol{q = -\frac{1}{3}}$

Method 2:

$y = p\sqrt{x} + \frac{q}{\sqrt{x}}$

$\frac{y}{\sqrt{x}} = p + \frac{q}{x}$ ⟵ Divide by $\sqrt{x}$ throughout.

$\frac{y}{\sqrt{x}} = \frac{q}{x} + p$

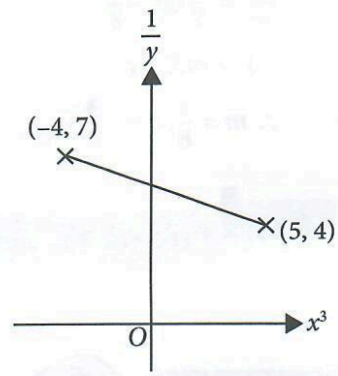
$Y = mX + c$

Since $m = q$ and $c = p$, then $\boldsymbol{p = 2}$ and $\boldsymbol{q = -\frac{1}{3}}$.

TRY TUTORIAL 8.1: Questions 3–5

WORKED EXAMPLE 3 Converting a linear form into a non-linear equation

The diagram shows part of a straight line graph of $\frac{1}{y}$ against x^3, passing through the points (–4, 7) and (5, 4). Express y in terms of x.



SOLUTION

$$\text{Gradient} = \frac{4-7}{5-(-4)} \longleftarrow \text{Gradient} \neq \frac{\frac{1}{4}-\frac{1}{7}}{5^3-(-4)^3}$$

$$= -\frac{1}{3}$$

$Y = mX + c$ — (1)

Substitute $m = -\frac{1}{3}$, $X = 5$ and $Y = 4$ into (1): $\longleftarrow X \neq 5^3, Y \neq \frac{1}{4}$

$$4 = -\frac{1}{3}(5)+c$$

$$4 = -\frac{5}{3}+c$$

$$c = \frac{17}{3}$$

Let $Y = \frac{1}{y}$ and $X = x^3$:

$$\frac{1}{y} = -\frac{1}{3}x^3+\frac{17}{3}$$

$$\frac{1}{y} = \frac{17-x^3}{3}$$

$$y = \frac{3}{17-x^3}$$

TRY TUTORIAL 8.1: Questions 6–10

NAME: ______________________ CLASS: ____________ DATE: ________

TUTORIAL 8.1

1. Express each of the following equations in the form $Y = mX + c$, where X and Y are functions of x and/or y, and m and c are constants. State the value of m and of c.

(a) $y = 6x^2 + 1$

(b) $\frac{y^3}{4} = 7 - x$

(c) $5x + \frac{1}{x} = 8y$

(d) $\sqrt{y} = \frac{1}{2x-9}$

(e) $y = 3(10^x)$

(f) $6e^{2y} = \sqrt[3]{x}$

2. In each of the following equations, a and b are constants. Express each equation in the form $Y = mX + c$, where X and Y are functions of x and/or y, and m and c are constants.

(a) $e^y = ab^x$

(b) $\sqrt{y} = \frac{ax}{x^2 - b}$

3. The variables x and y are related by the equation $py + qx = x^2$, where p and q are constants. When a graph of $\frac{y}{x}$ is plotted against x, the resulting line has a gradient of $\frac{1}{4}$ and an intercept of -7 on the $\frac{y}{x}$-axis. Find the value of p and of q.

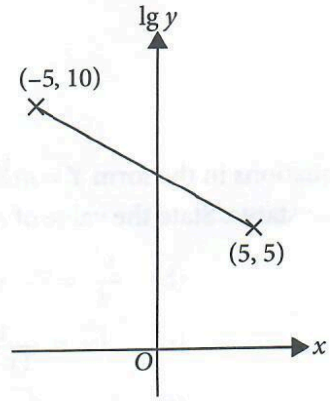
4. The variables x and y are related by an equation of the form $y = ax^b$, where a and b are constants. When a graph of $\ln y$ is plotted against $\ln x$, the resulting line has a gradient of -3 and a ($\ln y$)-intercept of $\frac{1}{2}$. Find the value of a and of b.

5. The variables x and y are related by the equation $y = 6e^{-x}$.

(i) Show that $\ln y = -x + \ln 6$.

(ii) Sketch the graph of $\ln y$ against x.

6.

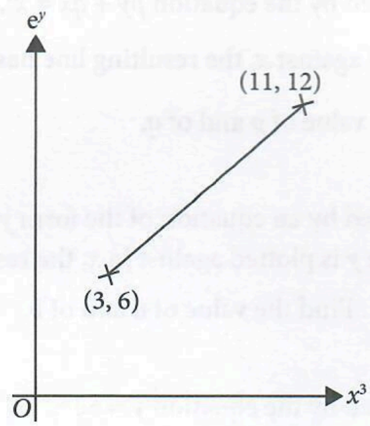


The diagram shows part of a straight line graph of lg y against x, passing through the points (−5, 10) and (5, 5).

(i) Express y in terms of x.

(ii) Find the value of x when lg $y = 4$.

7.



The diagram shows part of a straight line graph of e^y against x^3, passing through the points (3, 6) and (11, 12).

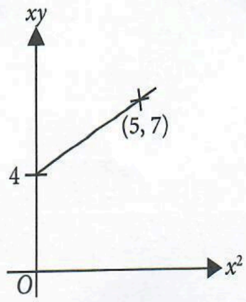
(i) Express y in terms of x.

(ii) Find the value of y when $x = 10$.

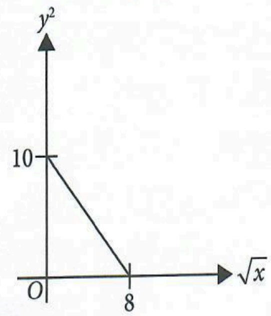
(iii) Another point, (h, k), lies on the graph of e^y against x^3. Give an example of the values of h and k.

8. Each of the following diagrams shows part of a straight line graph. Express y in terms of x.

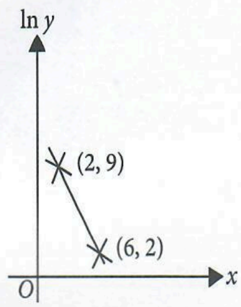
(a)



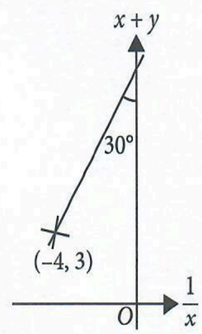
(b)



(c)



(d)



9. The variables x and y are related such that when values of $y - x^2$ are plotted against $\ln x$, a straight line is obtained. Given that the line passes through (1, 3) and (6, 8),

(i) express y in terms of x,

(ii) find the exact value of y when $x = 4$.

10. The variables x and y are such that when values of $\frac{1}{\sqrt{y}}$ are plotted against $\frac{1}{x}$, a straight line is obtained. It is given that $y = 0.25$ when $x = -1$ and that $y = 1$ when $x = 3$. Find the value of y when $x = 9$.

IDEAS

8.2 Applications of the Linear Law

Objectives Checklist

- Find the unknown constants of a non-linear equation from the corresponding straight line graph
- Model and solve problems in the sciences and in the real world using non-linear equations and their corresponding straight line graphs

Notes and Worked Examples

Non-linear equations and their corresponding straight line graphs can be used to **model** problems in the sciences and in the real world.

WORKED EXAMPLE 1 Solving problems involving the line of best fit

The table shows experimental values of two variables, x and y, which are connected by an equation of the form $yx^b = a$, where a and b are constants.

x	5	10	15	20
y	11.60	5.05	3.10	2.20

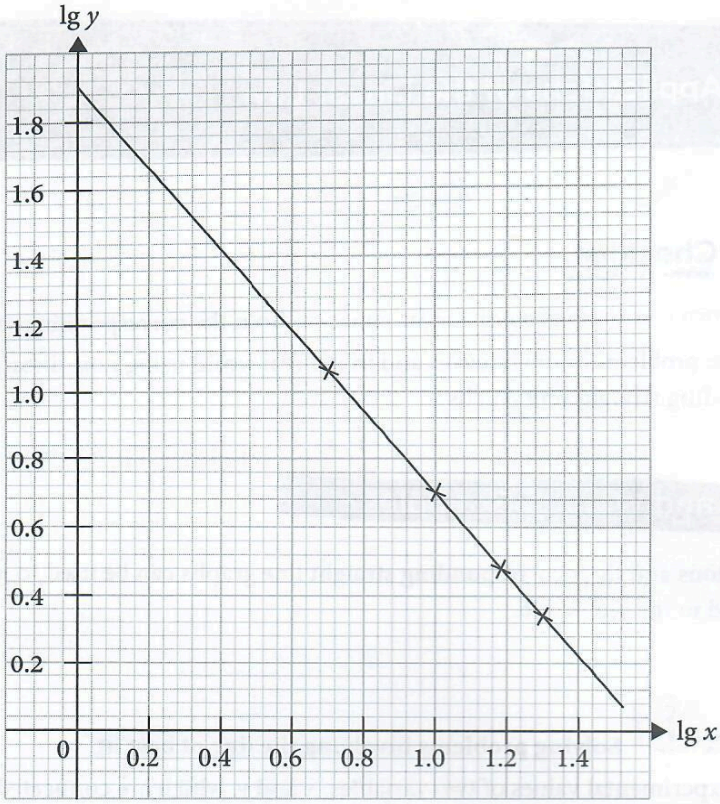
(i) Using a scale of 1 cm to 0.1 unit on each axis, plot lg y against lg x and draw a straight line graph.

(ii) Use your graph to estimate the value of a and of b.

SOLUTION

(i) Construct a table of values of lg x and lg y.

$X = \lg x$	0.699	1	1.176	1.301
$Y = \lg y$	1.064	0.703	0.491	0.342



(ii) $yx^b = a$

$\lg yx^b = \lg a$

$\lg y + b \lg x = \lg a$

$\lg y = -b \lg x + \lg a$ ⟵ Rearrange the equation into the form $Y = mX + c$.

From the graph, lg y-intercept = 1.9.

$\lg a = 1.9$

$a = 10^{1.9}$

$= 79$ (to 2 s.f.)

Using points (0, 1.9) and (1.2, 0.46),

gradient $= \dfrac{0.46-1.9}{1.2-0}$

$= -1.2$

$-b = -1.2$

$b = 1.2$

$\therefore$ **$a = 79$, $b = 1.2$**

TRY TUTORIAL 8.2: Questions 1, 2

WORKED EXAMPLE 2 **Applying the Linear Law in the sciences**

Focal length of a lens. In a science laboratory, a student is performing an experiment to determine the focal length, f mm, of a lens. He places an object at a distance, u mm, from the lens and records the distance, v mm, at which the image is seen on the other side of the lens. The table shows the results he obtains.

u	40	60	80	100	120
v	93	65	43	39	37

It is known that u, v and f are related by the equation $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$. It is believed that an error was made in recording one of the values of v.

(i) Plot $\frac{1}{v}$ against $\frac{1}{u}$ for the given data and draw a straight line graph.

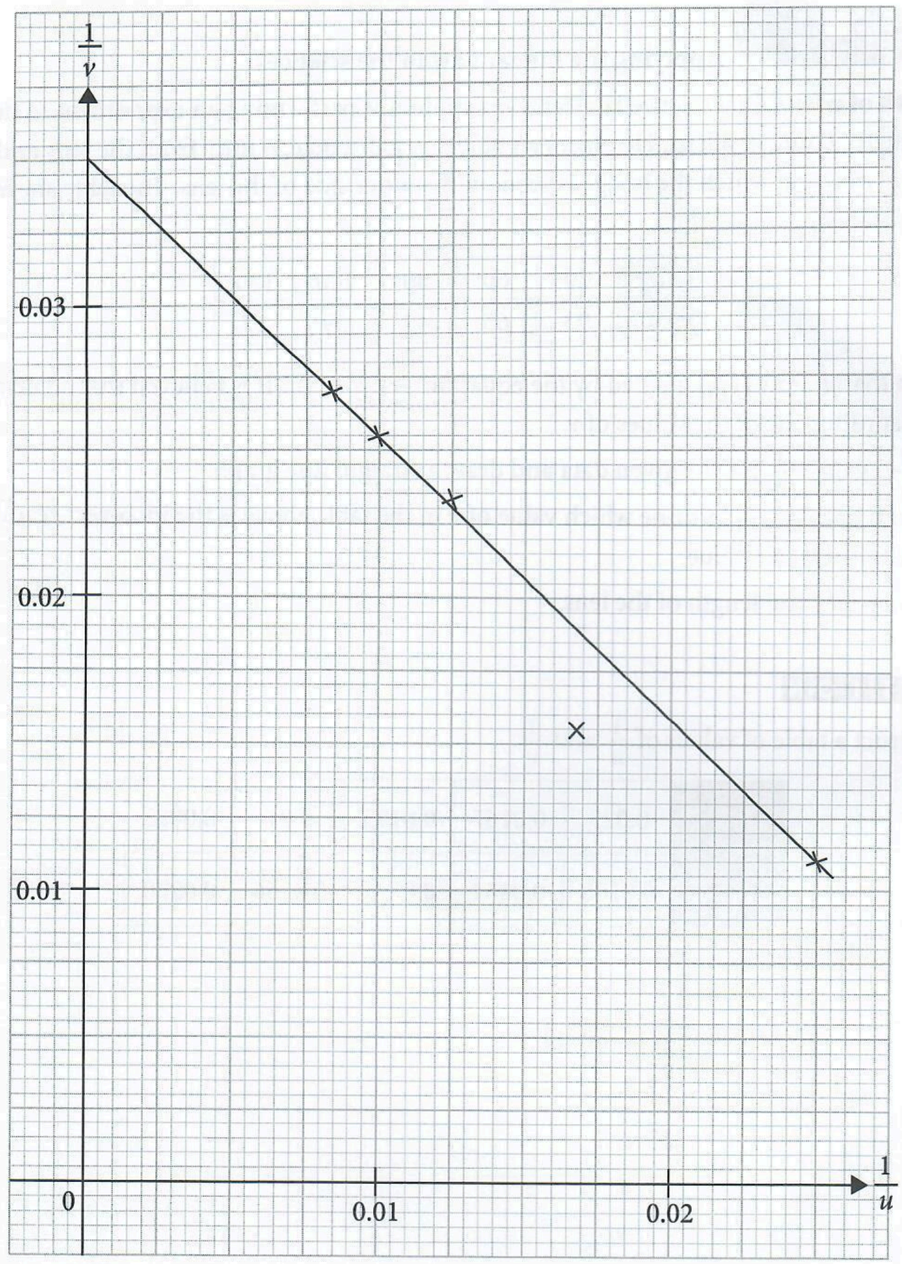
(ii) Use your graph to determine which value of v in the table above is the incorrect reading and estimate its correct value.

(iii) Estimate the focal length of the lens.

SOLUTION

(i) Construct a table of values of $\frac{1}{u}$ and $\frac{1}{v}$.

$X=\frac{1}{u}$	0.025	0.0167	0.0125	0.01	0.0083
$Y=\frac{1}{v}$	0.0108	0.0154	0.0233	0.0256	0.0270



(ii) From the graph, v = **65** is the incorrect reading.

Since $\frac{1}{v} = 0.019$, then $v = 53$ (to 2 s.f.)

$\therefore$ The correct value is v = **53**.

(iii) $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$

$\frac{1}{v}=-\frac{1}{u}+\frac{1}{f}$ ⟵ Rearrange the equation into the form $Y = mX + c$.

From the graph, $\frac{1}{v}$ -intercept = 0.035.

$$\frac{1}{f}=0.035$$

$$f = 29 \text{ (to 2 s.f.)}$$

∴ The focal length of the lens is about **29 mm**.

TRY TUTORIAL 8.2: Questions 3, 4

IDEAS

NAME: ______________________ CLASS: ____________ DATE: ________

TUTORIAL **8.2**

1. The table shows experimental values of two variables, x and y, which are related by the equation $y = ax^2 + bx$, where a and b are constants.

x	3	5	7	9
y	3.9	11.5	23.1	38.7

(i) Plot $\frac{y}{x}$ against x and draw a straight line graph.

(ii) Use your graph to estimate the value of a and of b.

2. The table shows experimental values of two variables, x and y, which are related by the equation $y = ax^b$, where a and b are constants.

x	5	10	15	20
y	29.8	84.3	154.9	238.5

(i) Express this equation in a form suitable for drawing a straight line graph.

(ii) Draw this graph and use it to estimate the value of a and of b.

(iii) Use the graph in part **(ii)** to solve the equation $ax^b = x^2$.

3. **Radioactivity of a compound.** Recorded values of the mass, m grams, of a radioactive substance, t hours after observations began, are shown in the table below.

***m* (grams)**	61.4	50.3	41.2	33.7
***t* (hours)**	10	20	30	40

It is known that m and t are related by the equation $m = m_0e^{-kt}$, where m_0 and k are constants.

(i) Plot ln m against t and draw a straight line graph.

Use your graph to estimate

(ii) the mass of the substance when the observations began,

(iii) the value of k,

(iv) the time taken for the substance to lose half of its original mass.

4. **Focal length of a lens.** In the end-of-year Physics practical examination, students are required to find the focal length, f cm, of an optical lens. Leo measures the object distance, u cm, and the corresponding image distance, v cm, and records the results in the table below.

u	30	40	50	60
v	30	24	21	20

It is known that u, v and f are related by the equation $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$. Leo claims that an error was made in recording one of the values of v.

(i) Plot $\frac{1}{v}$ against $\frac{1}{u}$ and hence identify the value of v in the table above that is the incorrect reading.

(ii) Draw the straight line graph and use it to estimate a value of v to replace the incorrect value of v found in part **(i)**.

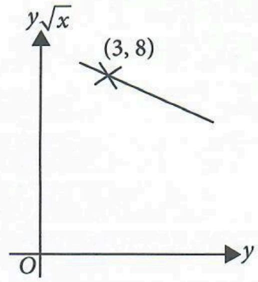
(iii) Estimate the value of f.

NAME: ______________________ CLASS: ____________ DATE: ____________

Quick Test 8

[Total marks: 15]

1.



The diagram shows part of a straight line graph drawn to represent the equation $y = \frac{a}{2\sqrt{x}+b}$, where a and b are constants. Given that the line passes through the point (3, 8) and has gradient $-\frac{1}{2}$, find the value of a and of b. [4]

2. **Speed of a toy truck.** The speed, v cm/s, of a toy truck, t s after passing a fixed point O, can be modelled by the equation $v = pe^{\frac{t}{q}} - 8$, where p and q are constants. The table below shows corresponding values of t and v.

t	2	4	6	8
v	30	24	19	15

(i) Using a scale of 1 cm to 1 unit on the t-axis and 10 cm to 1 unit on the $\ln(v + 8)$-axis, draw the graph of $\ln(v + 8)$ plotted against t. [3]

(ii) Use the graph to estimate the value of each of the constants p and q. [5]

(iii) State the speed of the toy truck at O. [1]

(iv) Explain how the graph could be used to find the value of t when the speed of the toy truck is 22 cm/s. [1]

(v) Using your values of p and q, calculate the value of t when the toy truck comes to rest. [1]

IDEAS